Operations on relations

Operations on relations

Universe of objects U, relations R, S on objects, set of objects X

Operations on relations		
Projection_1(R)	:	returns the set $\pi_1(R) = \{x \mid \exists y\ (x, y) \in R\}$.
Projection_2(R)	:	returns the set $\pi_2(R) = \{y \mid \exists x\ (x, y) \in R\}$.
Join(R, S)	:	returns the relation $R \circ S = \{(x, z) \mid \exists y \in X\ (x, y) \in R \wedge (y, z) \in S\}$
Post(X, R)	:	returns the set $post_R(X) = \{y \in U \mid \exists x \in X\ (x, y) \in R\}$.
Pre(X, R)	:	returns the set $pre_R(X) = \{y \in U \mid \exists x \in X\ (y, x) \in R\}$.

Encoding pairs

- Using automata to represent relations requires to encode pairs of objects.
- How should we encode a pair (n_1, n_2) of natural numbers?

Encoding pairs

- Assume n_1, n_2 are encoded by w_1, w_2 in the *lsbf* encoding
- Which should be the encoding of (n_1, n_2) ?
- Cannot be $w_1 w_2$, then the same word encodes different pairs
- First attempt: use a separator symbol $\&$, and encode (n_1, n_2) by $w_1 \& w_2$.
  - Problem: not even the identity relation is encoded as a regular language!

Encoding pairs

- Second attempt: encode (n_1, n_2) as a word over $\{0,1\} \times \{0,1\}$ (intuitively, the automaton reads w_1 and w_2 simultaneously).
  - Problem: what if w_1 and w_2 have different length?
  - Solution: fill the shortest one with 0s.

Example: the encoding of $(10, 35)$ is $\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$

  - We accept that the number k is encoded by all the words of $s_k 0^*$, where s_k is the *lsbf* encoding of k.
  - We call 0 the padding symbol or padding letter.

Encoding pairs

- So we assume:
  - The alphabet contains a padding letter #, different or not from the letters used to encode an object.
  - Each object x has a minimal encoding s_x .
  - The encodings of x are all the words of $s_x\#^*$.
  - A pair (x, y) of objects has a minimal encoding $s_{(x,y)}$.

  - The encodings of (x, y) are all the words of $s_{(x,y)}\#^*$.

Redefining acceptance

- Question: if objects (pairs of objects) are encoded by multiple words, which is the set of objects (pairs) recognized by a DFA or NFA?

  (We can no longer say: an object is recognized if ``its encoding'' is accepted by the DFA or NFA, because now there are multiple encodings)

- Question: because of the new definition of "set of objects recognized by an automaton", do we have to change the implementation of the set operations?

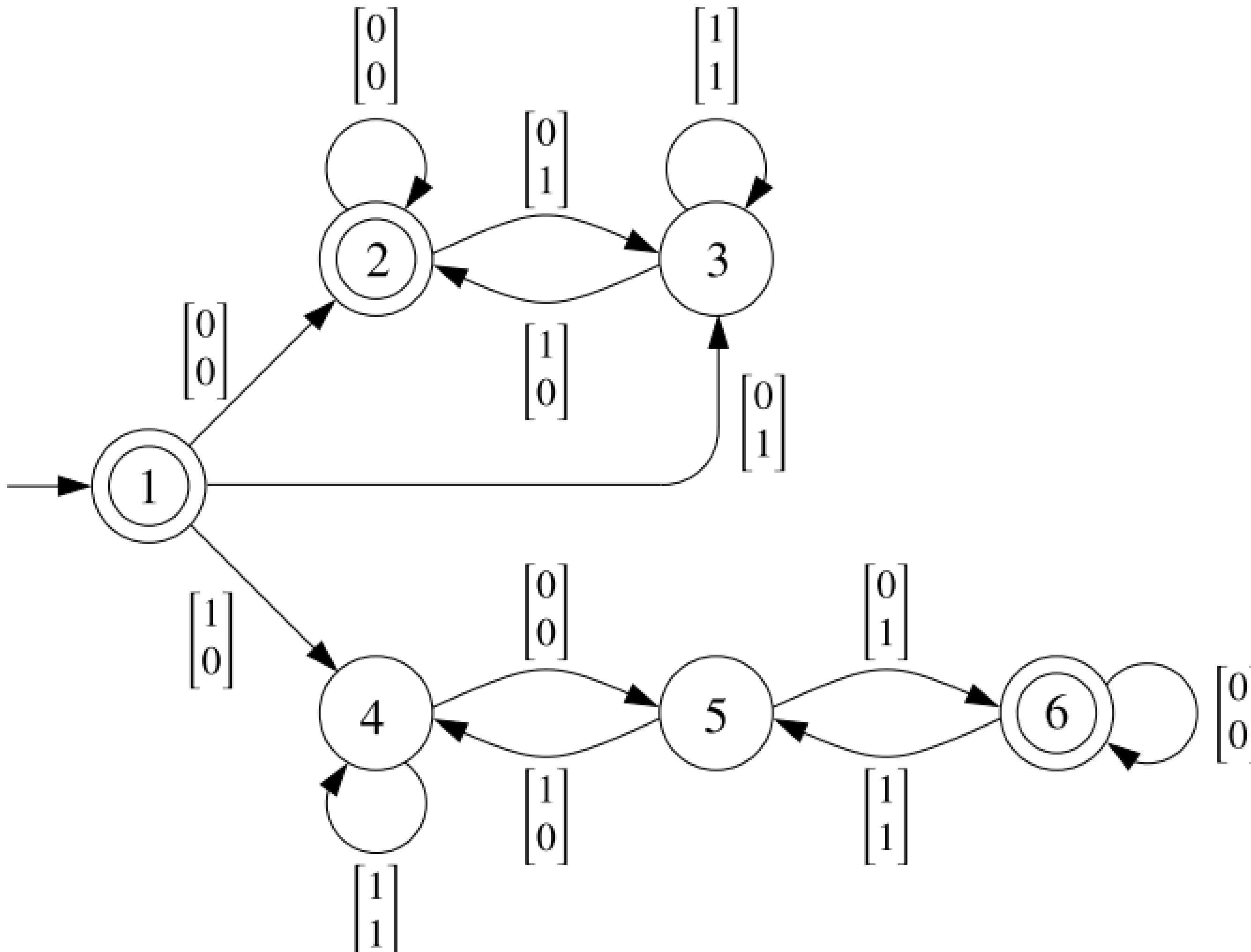
Redefining acceptance

- Definition: Assume an encoding of objects as words has been fixed. We say
  - An automaton accepts an object x if it accepts all encodings of x.
  - An automaton rejects an object x if it accepts no encoding of x.
  - An automaton recognizes a set of objects X if it accepts every object of X and rejects every other object.
- Observe: if an automaton accepts some, but not all the encodings of an object, then the automaton does not recognize any set. We say that such an automaton is ill formed. Automata that do recognize some set of objects are well formed.

Redefining acceptance

- The operations we have defined so far still work, in the following sense:
  - If the input(s) is (are) well formed, then the output is well formed
  - The output still satisfies the specification.
- Example: If A_1, A_2 are well formed NFAs recognizing sets of objects X_1 , X_2 then the automaton $A \coloneqq \mathrm{inter}(A_1, A_2)$ is well formed and recognizes $X_1 \cap X_2$.
  Proof of well formedness: If A recognizes an encoding w of an object x, then by definition of A both A_1 and A_2 recognize w. Since A_1 and A_2 are well formed they recognize all encodings of x, and so A also recognizes all encodings of x.

Transducers



Transducers

- A transducer over Σ is an NFA over the alphabet $\Sigma \times \Sigma$.
- We write $(a, b) \in \Sigma \times \Sigma$ as $\begin{bmatrix} a \\ b \end{bmatrix}$
- A transducer accepts a pair $(a_1 \dots a_n, b_1 \dots , b_n)$ of words if

  it accepts the word $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdots \begin{bmatrix} a_n \\ b_n \end{bmatrix}$.
- A transducer accepts a pair of objects if it accepts all ist encodings (which are pairs of words).
- A relation is regular if it is recognized by some transducer.

Examples of regular relations

- Examples of regular relations on numbers (*lsbf* encoding):
  - The identity relation $\{ (n, n) \mid n \in \mathbb{N} \}$
  - The relation $\{ (n, 2n) \mid n \in \mathbb{N} \}$
  - The relation $\{ (n, f(n)) \mid n \in \mathbb{N} \}$ where $f\colon \mathbb{N} \to \mathbb{N}$ is the Collatz function given by:

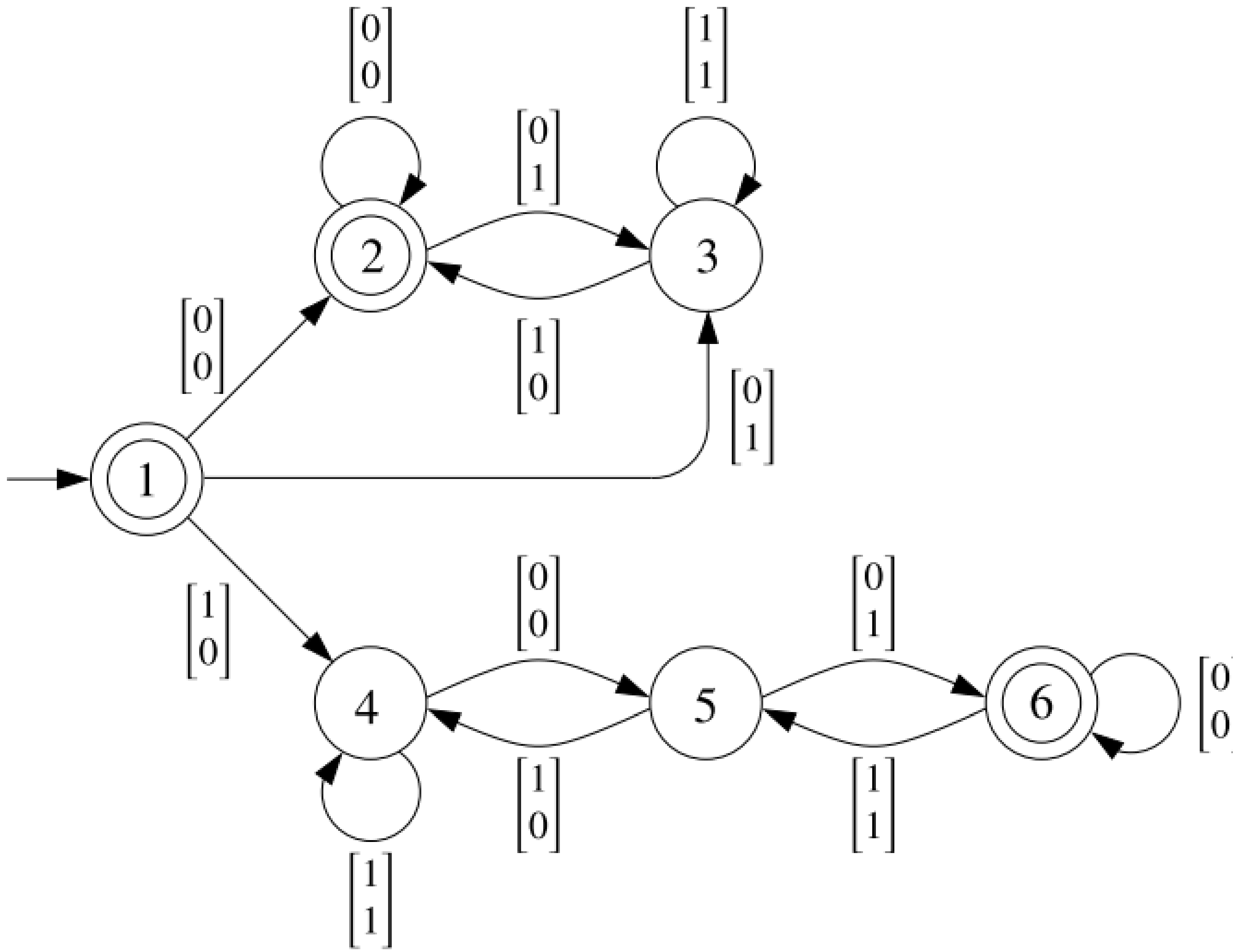
$$f(n) = \begin{cases} 3n + 1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

Deterministic transducers

- A transducer is deterministic if it is a DFA.
- Observe: if Σ has size n, then a state of a deterministic transducer with alphabet $\Sigma \times \Sigma$ has n^2 outgoing transitions.
- Warning! There is a different definition of determinism:
  - A letter $\begin{bmatrix} a \\ b \end{bmatrix}$ is interpreted as "output b on input a"
  - Deterministic transducer: only one move (and so only one output) for each input.

Implementing the operations

Computing projections



Computing projections

- Deleting the second component of all letters to compute the projection of a relation onto the first component is incorrect
  - Counterexample: $R = \{ (1,4) \}$
  - $s_{(4,1)} = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$
  - DFA for R:

Computing projections

- Problem: we may be accepting $s_x\ \#^k\ \#^*$ instead of $s_x\ \#^*$ and so according to the definition we are not acepting x !
- Solution: if after eliminating the second components some non-final state goes with $\#$ ... $\#$ to a final state, we mark the state as final.
- Complexity: linear in the size of the transducer
- Observe: the result of a projection may be a NFA, even if the transducer is deterministic.
- <u>This is the operation that prevents us from implementing all operations directly on DFAs.</u>

Computing projections

Proj_1(T)
Input: transducer $T = (Q, \Sigma \times \Sigma, \delta, Q_0, F)$
Output: NFA $A = (Q', \Sigma, \delta', Q'_0, F')$ with $\mathcal{L}(A) = \pi_1(\mathcal{L}(T))$

$Q' \leftarrow Q$; $Q'_0 \leftarrow Q_0$; $F'' \leftarrow F$
$\delta' \leftarrow \emptyset$
for all $(q, (a, b), q') \in \delta$ **do**
  add (q, a, q') **to** δ'
$F' \leftarrow$ *PadClosure*$((Q', \Sigma, \delta', Q'_0, F''), \#)$

PadClosure$(A, \#)$
Input: NFA $A = (\Sigma, Q, \delta, q_0, F)$
Output: new set F' of final states

$W \leftarrow F$; $F' \leftarrow \emptyset$
while $W \neq \emptyset$ **do**
  pick q **from** W
  add q **to** F'
  for all $(q', \#, q) \in \delta$ **do**
    if $q' \notin F'$ **then add** q' **to** W
return F'

Correctness

- Assume: transducer T recognizes a relation
- Prove: the projection automaton A recognizes a set, and this set is the projection onto the first component of the relation recognized by T.

a) A accepts either all encodings or no encoding of an object.

Assume A accepts at least one encoding w of an object x. We prove it accepts all.

If A accepts w , then T accepts $\begin{matrix} w \\ w' \end{matrix}$ for some w'.

By assumption T accepts $\begin{matrix} w \\ w' \end{matrix} \begin{bmatrix} \# \\ \# \end{bmatrix}^*$, and so A accepts $w\,\#^*$.

Moreover, $w = s_x \#^k$ for some $k > 0$, and so, by padding closure, A also accepts $s_x \#^j$ for every $j < k$.

Correctness

b) A only accepts words that are encodings of objects.
Follows easily from the fact that T satisfies the same property for pairs of objects.

c) If A accepts an object x, then T accepts (x, y) for some y.

x is accepted by A
$\Rightarrow$ s_x is accepted by A (part a))
$\Rightarrow$ $\begin{matrix} s_x \\ w \end{matrix}$ is accepted by T for some w

By assumption, T only accepts pairs of words encoding some pair of objects. So w encodes some object y. By assumption, T then accepts all encodings of (x, y). So T accepts (x, y).

Correctness

d) If a pair of objects (x, y) is accepted by T, then x is accepted by A.

(x, y) is accepted by T

$\Rightarrow$ w_x w_y is accepted by T for some encodings w_x, w_y of x and y

$\Rightarrow$ w_x is accepted by A

$\Rightarrow$ x is accepted by A (part a))

Computing joins

- Goal: given transducers T_1, T_2 recognizing relations R_1, R_2, construct a transducer $T_1 \circ T_2$ recognizing the relation $R_1 \circ R_2$.
- First step: construct a transducer T that accepts $\begin{matrix} w \\ v \end{matrix}$ iff there is a "connecting" word u such that $\begin{matrix} w \\ u \end{matrix}$ is accepted by T_1 and $\begin{matrix} u \\ v \end{matrix}$ is accepted by T_2.
- We slightly modify the pairing construction.

Computing joins

Pairing construction

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \xrightarrow{a} \begin{bmatrix} q_1' \\ q_2' \end{bmatrix} \quad \text{iff} \quad \begin{array}{l} q_1 \xrightarrow{a} q_1' \\ q_2 \xrightarrow{a} q_2' \end{array}$$

Join construction

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \xrightarrow{\begin{bmatrix} a \\ b \end{bmatrix}} \begin{bmatrix} q_1' \\ q_2' \end{bmatrix} \quad \text{iff} \quad \begin{array}{l} q_1 \xrightarrow{\begin{bmatrix} a \\ c \end{bmatrix}} q_1' \\ q_2 \xrightarrow{\begin{bmatrix} c \\ b \end{bmatrix}} q_2' \end{array}$$

for some $c \in \Sigma$

Computing joins

- With the join construction, transducer T has a run

$$\begin{bmatrix} q_{01} \\ q_{02} \end{bmatrix} \xrightarrow{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \xrightarrow{\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \cdots \begin{bmatrix} q_{(n-1)1} \\ q_{(n-1)2} \end{bmatrix} \xrightarrow{\begin{bmatrix} a_n \\ b_n \end{bmatrix}} \begin{bmatrix} q_{n1} \\ q_{n2} \end{bmatrix}$$

iff T_1 and T_2 have runs

$$q_{01} \xrightarrow{\begin{bmatrix} a_1 \\ c_1 \end{bmatrix}} q_{11} \xrightarrow{\begin{bmatrix} a_2 \\ c_2 \end{bmatrix}} q_{21} \cdots q_{(n-1)1} \xrightarrow{\begin{bmatrix} a_n \\ c_n \end{bmatrix}} q_{n1}$$

$$q_{02} \xrightarrow{\begin{bmatrix} c_1 \\ b_1 \end{bmatrix}} q_{12} \xrightarrow{\begin{bmatrix} c_2 \\ b_2 \end{bmatrix}} q_{22} \cdots q_{(n-1)2} \xrightarrow{\begin{bmatrix} c_n \\ b_n \end{bmatrix}} q_{n2}$$

Computing joins

- Second step: We have the same problem as before.
  - Let $R_1 = \{ (2,4) \}$, $R_2 = \{ (4,2) \}$.
    Then $R_1 \circ R_2 = \{ (2,2) \}$.
  - But the operation we have just defined does not yield the correct result.
  - Solution: apply the padding closure again with padding symbol $\begin{bmatrix} \# \\ \# \end{bmatrix}$.

Computing joins

$Join(T_1, T_2)$

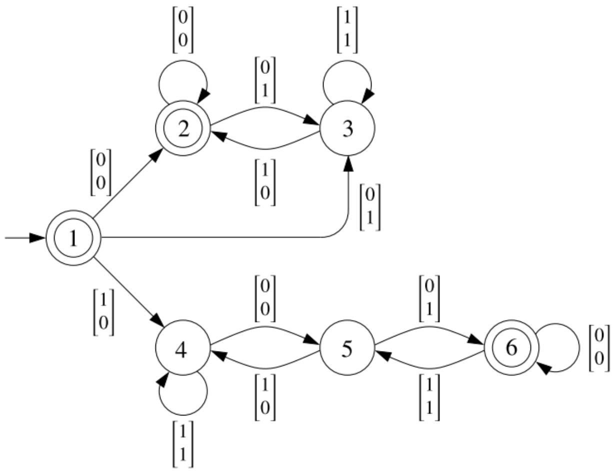
Input: transducers $T_1 = (Q_1, \Sigma \times \Sigma, \delta_1, Q_{01}, F_1)$,
$T_2 = (Q_2, \Sigma \times \Sigma, \delta_2, Q_{02}, F_2)$

Output: transducer $T_1 \circ T_2 = (Q, \Sigma \times \Sigma, \delta, Q_0, F)$

$Q, \delta, F' \leftarrow \emptyset;\ Q_0 \leftarrow Q_{01} \times Q_{02}$
$W \leftarrow Q_0$
while $W \neq \emptyset$ **do**
  pick $[q_1, q_2]$ **from** W
  add $[q_1, q_2]$ **to** Q
  if $q_1 \in F_1$ and $q_2 \in F_2$ **then add** $[q_1, q_2]$ **to** F'
  for all $(q_1, (a, c), q_1') \in \delta_1, (q_2, (c, b), q_2') \in \delta_2$ **do**
    add $([q_1, q_2], (a, b), [q_1', q_2'])$ **to** δ
    if $[q_1', q_2'] \notin Q$ **then add** $[q_1', q_2']$ **to** W
$F \leftarrow$ **PadClosure**$((Q, \Sigma \times \Sigma, \delta, Q_0, F'), (\#, \#))$

Computing joins

- Example:
  - Let f be the Collatz function.
  - Let $R_1 = R_2 = \{ (n, f(n)) \mid n \geq 0 \}$.
  - Then $R_1 \circ R_2 = \{ (n, f(f(n))) \mid n \geq 0 \}$.



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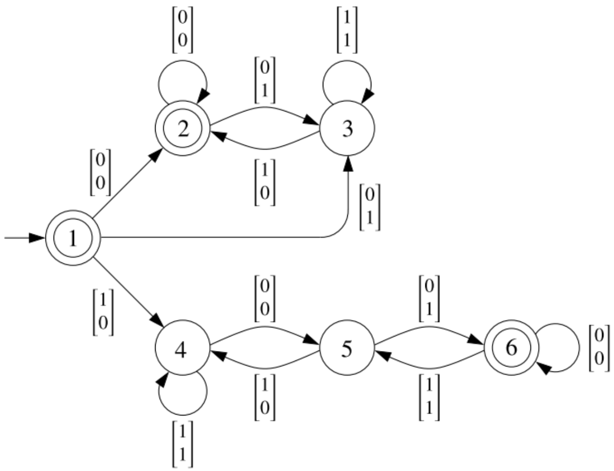

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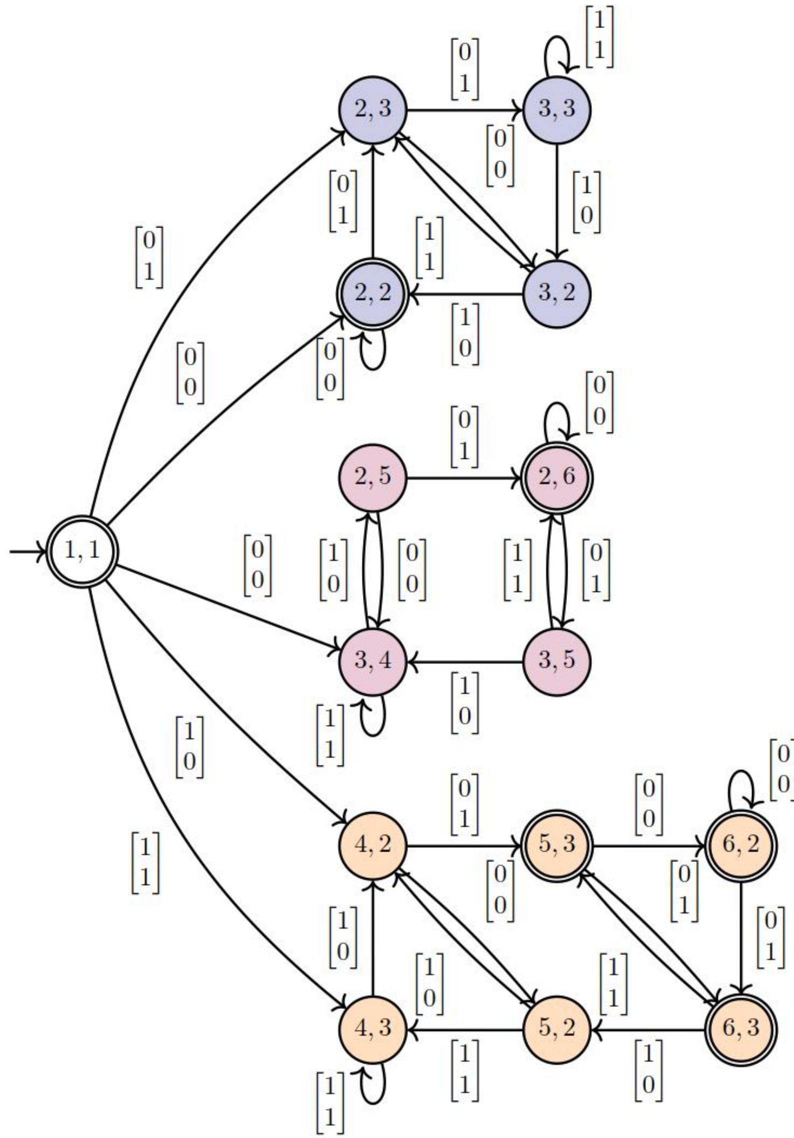
Computing joins



Computing Pre and Post

- Goal (for Post): given
  - an automaton A recognizing a set X, and
  - a transducer T recognizing a relation R

  construct an automaton B recognizing the set

  $$\text{Post}(X, R) = \{ y \mid \exists\, x \in X : (x, y) \in R \}$$

  We slightly modify the construction for join.

Computing Pre and Post

Join construction

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \xrightarrow{\begin{bmatrix} a \\ b \end{bmatrix}} \begin{bmatrix} q_1' \\ q_2' \end{bmatrix}$$

iff

$$q_1 \xrightarrow{\begin{bmatrix} a \\ \color{red}{c} \end{bmatrix}} q_1'$$

$$q_2 \xrightarrow{\begin{bmatrix} \color{red}{c} \\ b \end{bmatrix}} q_2'$$

for some $\color{red}{c} \in \Sigma$

Post construction

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} q_1' \\ q_2' \end{bmatrix}$$

iff

$$q_1 \xrightarrow{\color{red}{a}} q_1'$$

$$q_2 \xrightarrow{\begin{bmatrix} \color{red}{a} \\ b \end{bmatrix}} q_2'$$

for some $\color{red}{a} \in \Sigma$

Computing Pre and Post

Join(T_1, T_2)

Input: transducers $T_1 = (Q_1, \Sigma \times \Sigma, \delta_1, Q_{01}, F_1)$,
$T_2 = (Q_2, \Sigma \times \Sigma, \delta_2, Q_{02}, F_2)$

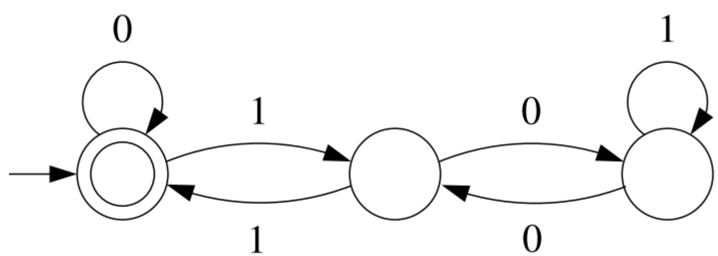
Output: transducer $T_1 \circ T_2 = (Q, \Sigma \times \Sigma, \delta, Q_0, F)$

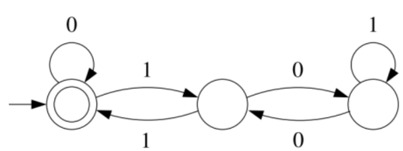
$Q, \delta, F' \leftarrow \emptyset$; $Q_0 \leftarrow Q_{01} \times Q_{02}$
$W \leftarrow Q_0$
while $W \neq \emptyset$ **do**
  pick $[q_1, q_2]$ **from** W
  add $[q_1, q_2]$ **to** Q
  if $q_1 \in F_1$ and $q_2 \in F_2$ **then add** $[q_1, q_2]$ **to** F'
  for all $(q_1, (a, c), q'_1) \in \delta_1, (q_2, (c, b), q'_2) \in \delta_2$ **do**
    add $([q_1, q_2], (a, b), [q'_1, q'_2])$ **to** δ
    if $[q'_1, q'_2] \notin Q$ **then add** $[q'_1, q'_2]$ **to** W
$F \leftarrow$ **PadClosure**$((Q, \Sigma \times \Sigma, \delta, Q_0, F'), (\#, \#))$

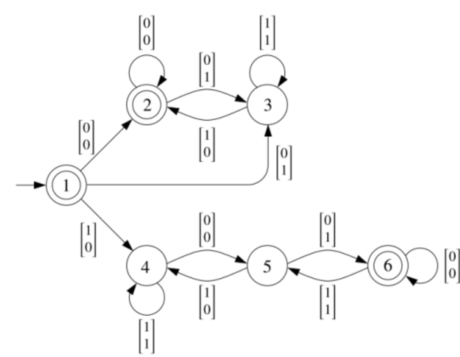
  for all $(q_1, (a, c), q'_1) \in \delta_1, (q_2, c, q'_2) \in \delta_2$ **do**
    add δ **to** $([q_1, q_2], a, [q'_1, q'_2])$

Computing Pre and Post

- Example.
  - Let f be the Collatz function.
  - We compute the set $\{f(n) \mid n \text{ is a multiple of } 3\}$
- DFA for the multiples of 3 in *lsfb* encoding



Post(

, 

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Computing Pre and Post