Operations and tests on sets: Implementation on DFAs

Operations and tests

Universe of objects U, sets of objects X, Y, object x.

Operations on sets		
Complement(X)	:	returns $U \setminus X$.
Intersection(X, Y)	:	returns $X \cap Y$.
Union(X, Y)	:	returns $X \cup Y$.

Tests on sets		
Member(x, X)	:	returns **true** if $x \in X$, **false** otherwise.
Empty(X)	:	returns **true** if $X = \emptyset$, **false** otherwise.
Universal(X)	:	returns **true** if $X = U$, **false** otherwise.
Included(X, Y)	:	returns **true** if $X \subseteq Y$, **false** otherwise.
Equal(X, Y)	:	returns **true** if $X = Y$, **false** otherwise.

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- Membership: trivial algorithm, linear in the length of the word.
- Complement: exchange final and non-final states. Linear (or even constant) time.
- Generic implementation of binary boolean operations based on pairing.

Pairing

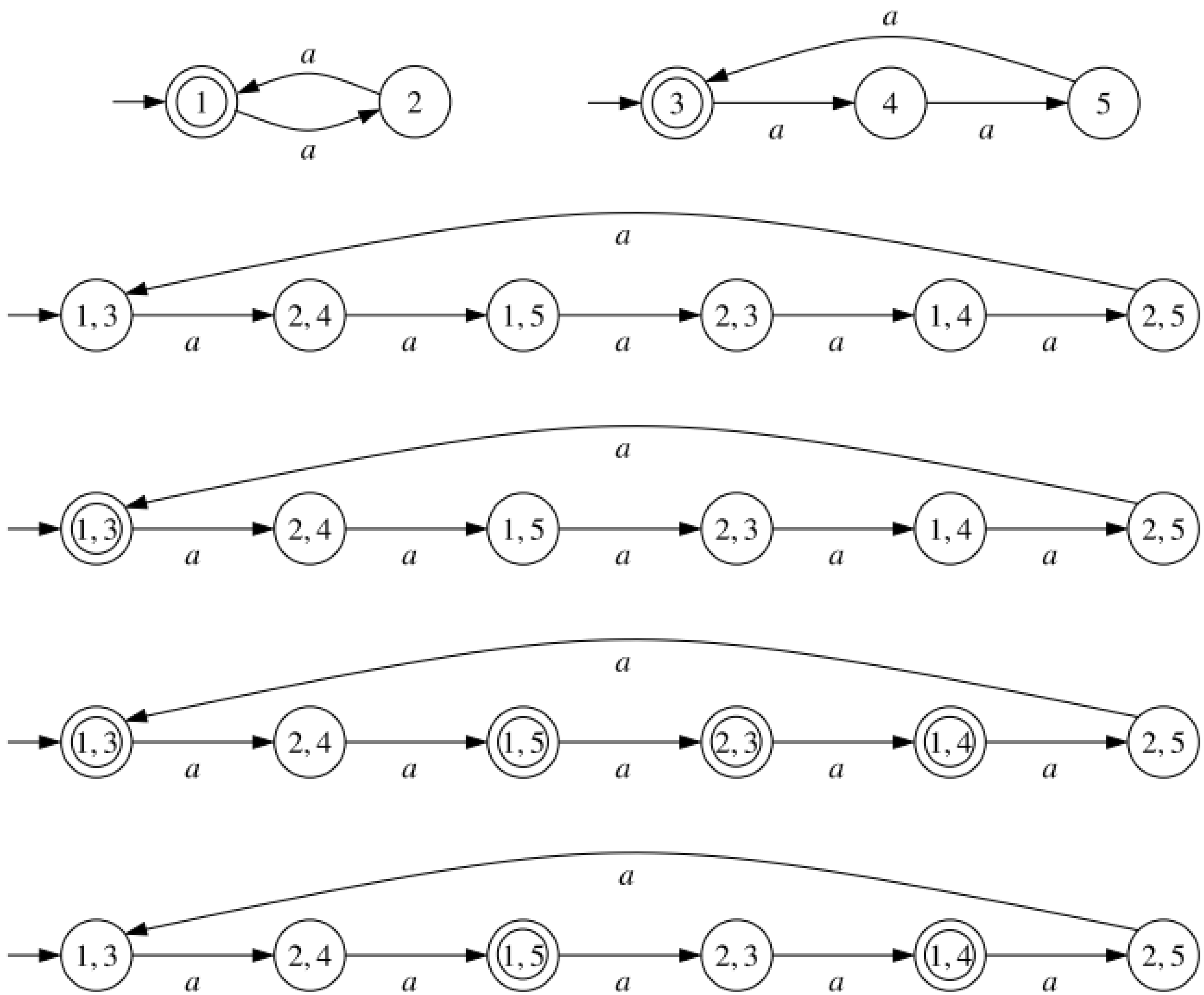
Definition. Let $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ be DFAs.

The pairing $[A_1, A_2]$ of A_1 and A_2 is the tuple (Q, Σ, δ, q_0) where

- $Q = \{ [q_1, q_2] \mid q_1 \in Q_1, q_2 \in Q_2\}$
- $\delta = \{ ([q_1, q_2], a, [q_1', q_2']) \mid (q_1, a, q_1') \in \delta_1, (q_2, a, q_2') \in \delta_2 \}$
- $q_0 = [q_{01}, q_{02}]$

The run of $[A_1, A_2]$ on a word of Σ^* is defined as for DFAs

Pairing



Pairing

- Another example: DFA for the language of words with an even number of as and even number of bs (and even number of cs ...).

Generic algorithm for binary boolean operations

- We assign to a binary boolean operator $\odot$ an operation on languages $\widehat{\odot}$ as follows:

$$L_1 \mathbin{\widehat{\odot}} L_2 = \{\ w \in \Sigma^* \mid (w \in L_1) \odot (w \in L_2)\ \}$$

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- For example:

Language operation	$b_1 \odot b_2$
Union	$b_1 \vee b_2$
Intersection	$b_1 \wedge b_2$
Set difference ($L_1 \setminus L_2$)	$b_1 \wedge \neg b_2$
Symmetric difference ($L_1 \setminus L_2 \cup L_2 \setminus L_1$)	$b_1 \Leftrightarrow \neg b_2$

Generic algorithm for binary boolean operations

$BinOp[\odot](A_1, A_2)$

Input: DFAs $A_1 = (Q_1, \Sigma, \delta_1, Q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, Q_{02}, F_2)$

Output: DFA $A = (Q, \Sigma, \delta, Q_0, F)$ with $L(A) = L(A_1) \mathbin{\widehat{\odot}} L(A_2)$

$Q, \delta, F \leftarrow \emptyset$

$q_0 \leftarrow [q_{01}, q_{02}]$

$W \leftarrow \{q_0\}$

while $W \neq \emptyset$ **do**

    pick $[q_1, q_2]$ **from** W

    add $[q_1, q_2]$ **to** Q

    if $(q_1 \in F_1) \odot (q_2 \in F_2)$ **then add** $[q_1, q_2]$ **to** F

    for all $a \in \Sigma$ **do**

        $q'_1 \leftarrow \delta_1(q_1, a)$; $q'_2 \leftarrow \delta_2(q_2, a)$

        if $[q'_1, q'_2] \notin Q$ **then add** $[q'_1, q'_2]$ **to** W

        add $([q_1, q_2], a, [q'_1, q'_2])$ **to** δ

Generic algorithm for binary boolean operations

- Complexity: the pairing of DFAs with n_1 and n_2 states has $O(n_1 \cdot n_2)$ states.
- Hence: for DFAs with n_1 and n_2 states over an alphabet with k letters, binary operations can be computed in $O(k \cdot n_1 \cdot n_2)$ time.
- Further: there is a family of languages for which the computation of intersection takes $\Theta(k \cdot n_1 \cdot n_2)$ time.

Language tests

- Emptiness: a DFA is empty iff it has no final states
- Universality: a DFA is universal iff it has only final states
- Inclusion: $L_1 \subseteq L_2$ iff $L_1 \setminus L_2 = \emptyset$
- Equality: $L_1 = L_2$ iff $(L_1 \setminus L_2) \cup (L_2 \setminus L_1) = \emptyset$

Inclusion test

InclDFA(A_1, A_2)

Input: DFAs $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$

Output: true if $L(A_1) \subseteq L(A_2)$, **false** otherwise

$Q \leftarrow \emptyset$;

$W \leftarrow \{[q_{01}, q_{02}]\}$

while $W \neq \emptyset$ **do**

  pick $[q_1, q_2]$ **from** W

  add $[q_1, q_2]$ **to** Q

  if $(q_1 \in F_1)$ **and** $(q_2 \notin F_2)$ **then return false**

  for all $a \in \Sigma$ **do**

    $q'_1 \leftarrow \delta_1(q_1, a)$; $q'_2 \leftarrow \delta_2(q_2, a)$

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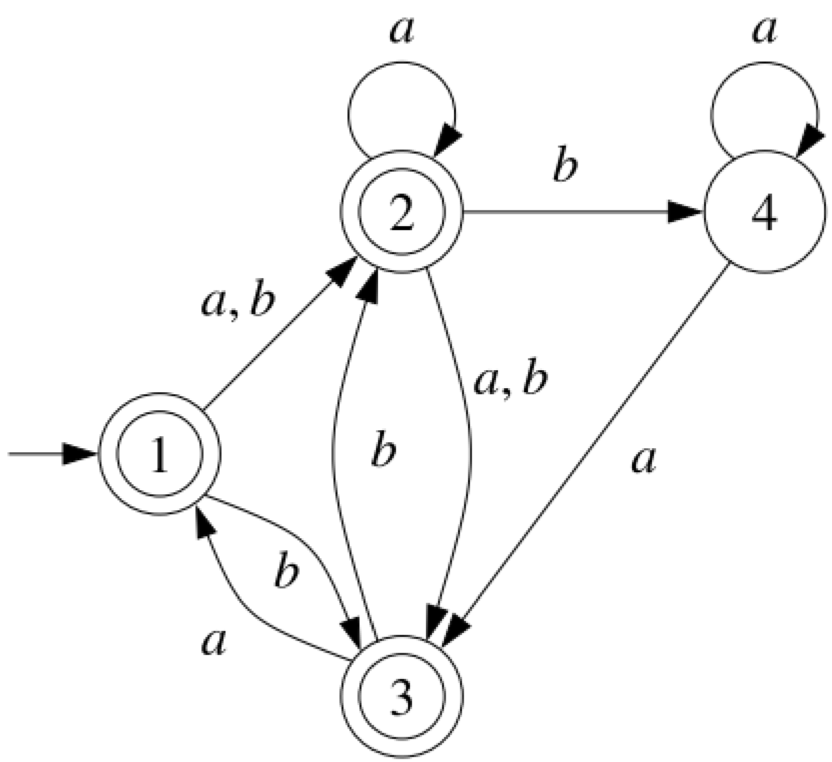
return true

Operations and tests on sets: Implementation on NFAs

Membership



Prefix read	W
ϵ	$\{1\}$
a	$\{2\}$
aa	$\{2, 3\}$
aaa	$\{1, 2, 3\}$
$aaab$	$\{2, 3, 4\}$
$aaabb$	$\{2, 3, 4\}$
$aaabba$	$\{1, 2, 3, 4\}$

Membership

$MemNFA[A](w)$

Input: NFA $A = (Q, \Sigma, \delta, Q_0, F)$, word $w \in \Sigma^*$,

Output: **true** if $w \in \mathcal{L}(A)$, **false** otherwise

$W \leftarrow Q_0$;
while $w \neq \varepsilon$ **do**
  $U \leftarrow \emptyset$
  for all $q \in W$ **do**
    add $\delta(q, head(w))$ **to** U
  $W \leftarrow U$
  $w \leftarrow tail(w)$
return $(W \cap F \neq \emptyset)$

Complexity:

- While loop executed $|w|$ times
- For loop executed at most $|Q|$ times
- Each execution of the loop body takes $O(|Q|)$ time

- Overall: $O(|Q|^2 \cdot |w|)$ time

Complement

- Swapping final and non-final states does not work
- Solution: determinize <u>and then</u> swap states
- Problem: Exponential blow-up in size!!

To be avoided whenever possible!!

- No better way: there are NFAs with n states such that the smallest NFA for their complement has $\Theta(2^n)$ states.

Complement

Let $\Sigma = \{a, b\}$. For every $n \geq 1$, let L_n be the language of the regular expression

$$\Sigma^*(a\Sigma^{n-1}b + b\Sigma^{n-1}a)\Sigma^*$$

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Proposition: For every $n \geq 1$, every NFA for $\overline{L_n}$ has at least 2^n states.

Proof. Observe: $ww \in \overline{L_n}$
for every $w \in \Sigma^n$.

Take an arbitrary NFA for $\overline{L_n}$.

For every $w \in \Sigma^n$ let q_u be the
state reached after reading w
in an accepting run of ww.

For every $w, v \in \Sigma^n$ we have:
$w \neq v \implies q_w \neq q_v$

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Union and intersection

- The pairing construction still works for intersection, with the same complexity.
- It also works for union, but only if the NFAs are complete, i.e., they have at least one run for each word.
- Optimal construction for intersection (same example as for DFAs).
- Non-optimal construction for union. There is another construction which produces an NFA with $|Q_1| + |Q_2|$ states, instead of $|Q_1| \cdot |Q_2|$: just put the automata side by side!

Intersection

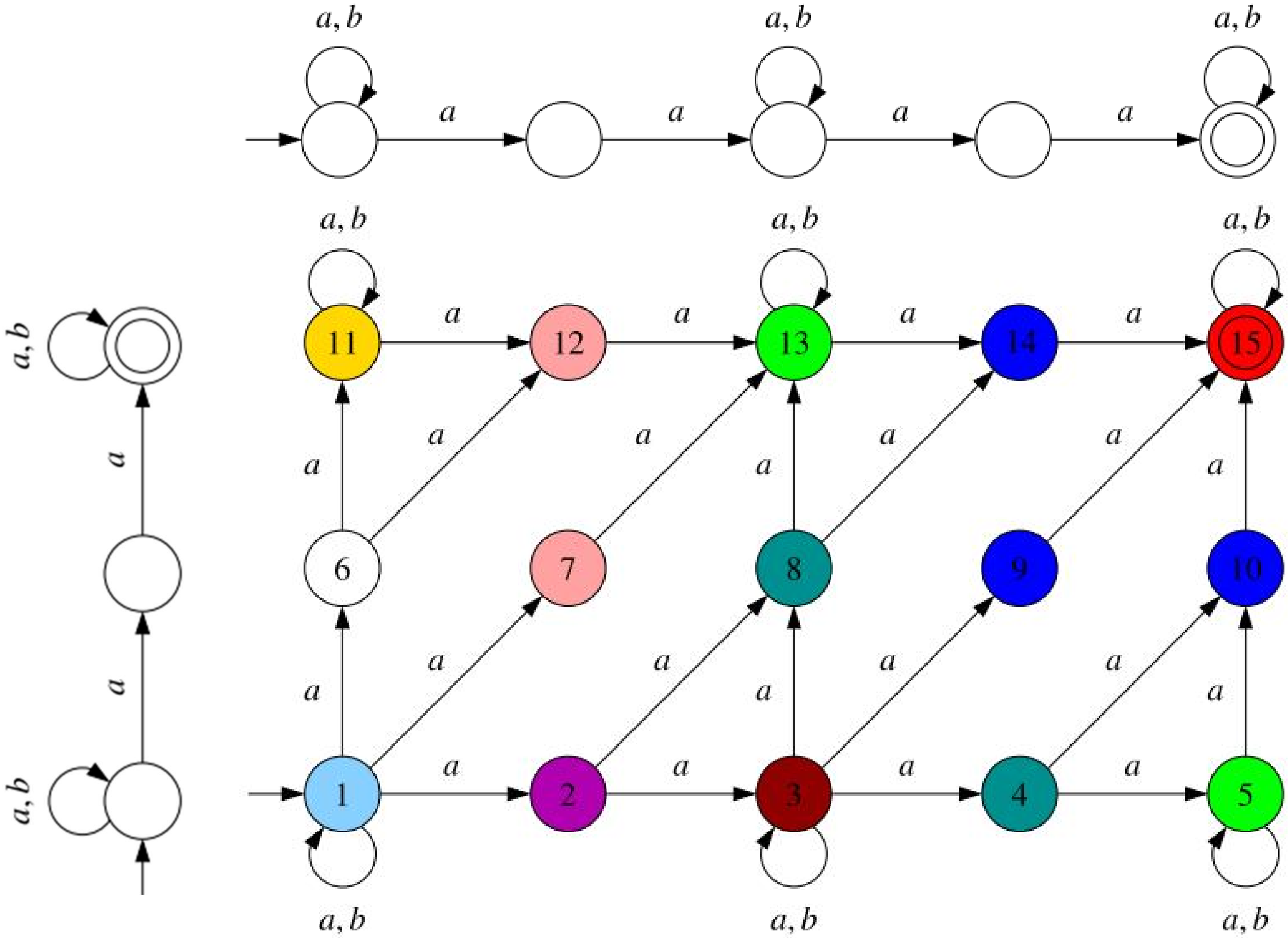
IntersNFA(A_1, A_2)

Input: NFA $A_1 = (Q_1, \Sigma, \delta_1, Q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, Q_{02}, F_2)$

Output: NFA $A_1 \cap A_2 = (Q, \Sigma, \delta, Q_0, F)$ with $L(A_1 \cap A_2) = L(A_1) \cap L(A_2)$

$Q, \delta, F \leftarrow \emptyset$; $Q_0 \leftarrow Q_{01} \times Q_{02}$
$W \leftarrow Q_0$
while $W \neq \emptyset$ **do**
    pick $[q_1, q_2]$ **from** W
    add $[q_1, q_2]$ **to** Q
    if $(q_1 \in F_1)$ **and** $(q_2 \in F_2)$ **then add** $[q_1, q_2]$ **to** F
    for all $a \in \Sigma$ **do**
        for all $q'_1 \in \delta_1(q_1, a), q'_2 \in \delta_2(q_2, a)$ **do**
            if $[q'_1, q'_2] \notin Q$ **then add** $[q'_1, q'_2]$ **to** W
            add $([q_1, q_2], a, [q'_1, q'_2])$ **to** δ

Intersection



Emptiness and Universality

- Like DFAs, an NFA is empty iff every state is non-final.
- However, contrary to DFAs, it does not hold that an NFA is universal iff every state is final. Both directions fail!
- Emptiness is decidable in linear time.
- Universality is PSPACE-complete.

Crash course on PSPACE

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- NPSPACE: Class of decision problems for which there is a nondeterministic algorithm that
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- Savitch´s theorem: PSPACE = NPSPACE

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- PSPACE-complete: A problem is PSPACE-complete if
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- PSPACE-complete problems:
  - Acceptance of linearly bounded automata (LBA): Given a LBA, i.e., a deterministic Turing machine M that only visits the cell tapes occupied by the input, and an input x, does M accept x ?
  - QBF: Is a given quantified boolean formula true?

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So it suffices to give a nondeterministic algorithm that, given an NFA A as input:

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- has at least one terminating execution answering „non-universal" if A is not universal, and
- only uses polynomial memory in the size of the input.

The algorithm guesses a word letter by letter, simulating the run of the equivalent DFA on it, and stops if at some point the state of the DFA is non-final.

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By reduction from the acceptance problem for LBA.

- Let M be a LBA, let x be an input for M. We construct in polynomial time a NFA A such that

$$M \text{ accepts } x \quad \text{iff} \quad A \text{ is not universal}$$

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- Configuration of M: sequence of the form $a_1 a_2 \cdots a_i \; q \; a_{i+1} \cdots a_n$ where $a_1, a_2, \ldots, a_n \in \Sigma$, $n = |x|$, $q \in Q$.

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- Encode the run of M on x as a word $w = c_0 \# c_1 \# \cdots \# c_n$ where each c_i encodes a configuration of M and c_o is the initial configuration for x.

Universality is PSPACE complete

- Idea: construct A so that it accepts all words that are not the encoding of an accepting run of M on x. Then
  - if M accepts x then A accepts all words but w
    $\Longrightarrow A$ is not universal
  - if M rejects x then A accepts all words
    $\Longrightarrow A$ is universal

Universality is PSPACE complete

- The run of M on x is the unique word satisfying the following three properties:
  1. w is a sequence of configurations separated by $\#$
  2. w starts with the initial configuration of M on x
  3. every configuration in w is followed by the successor configuration of M
- Further, the run is accepting iff
  4. w ends with a final configuration of M

Universality is PSPACE complete

- We construct NFAs $A_1, \dots, A_4$ with polynomially many states recognizing
  1. All words that **do not** consist of a sequence of configurations separated by $\#$
  2. All words that **do not** start with the initial configuration of M on x
  3. All words in which some configuration **is not** followed by the successor configuration
  4. All words that **do not** end with a final configuration of M
- Let A be a NFA recognizing $L(A_1) \cup L(A_2) \cup L(A_3) \cup L(A_4)$

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Deciding universality of NFAs

- Complement and check for emptiness
  - Needs exponential time and space.
- Improvements:
  - Check for emptiness while complementing (on-the-fly check).
  - Subsumption test.

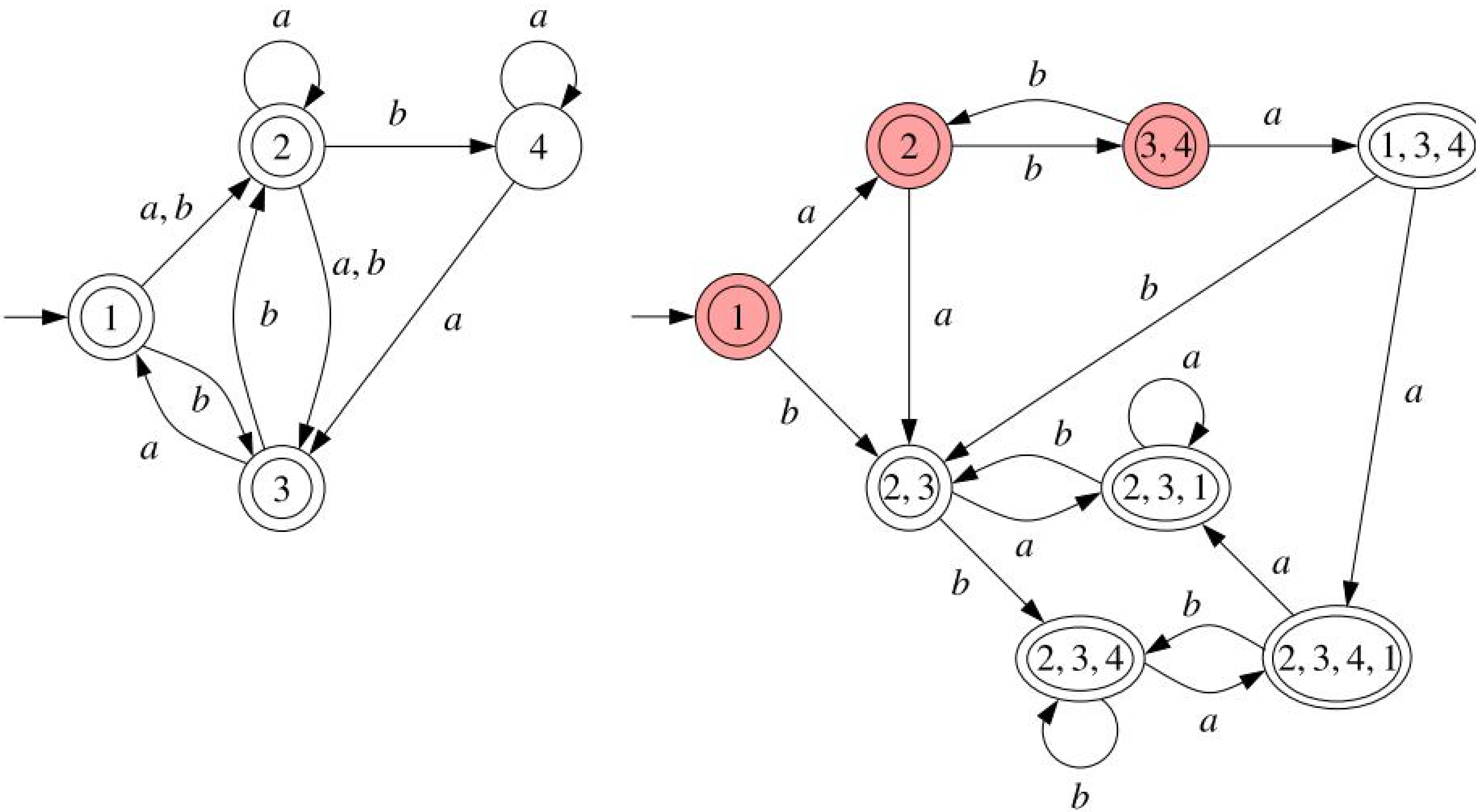
Subsumption test

- Let A be an NFA and let $B = NFAtoDFA(A)$. A state Q' of B is minimal if no other state Q'' satisfies $Q'' \subset Q'$.
- Proposition: A is universal iff every minimal state of B is final.

  Proof:

  A is universal
  iff B is universal
  iff every state of B is final
  iff every state of B contains a final state of A
  iff every minimal state of B contains a final state of A
  iff every minimal state of B is final

Subsumption test



Subsumption test

UnivNFA(A)

Input: NFA $A = (Q, \Sigma, \delta, Q_0, F)$

Output: **true** if $L(A) = \Sigma^*$, **false** otherwise

$\mathcal{Q} \leftarrow \emptyset$;
$\mathcal{W} \leftarrow \{ \{q_0\} \}$
while $\mathcal{W} \neq \emptyset$ **do**
  pick Q' **from** $\mathcal{W}$
  if $Q' \cap F = \emptyset$ **then return false**
  add Q' **to** $\mathcal{Q}$
  for all $a \in \Sigma$ **do**
    if $\mathcal{W} \cup \mathcal{Q}$ contains no $Q'' \subseteq \delta(Q', a)$ **then add** $\delta(Q', a)$ **to** $\mathcal{W}$
return true

Subsumption test

- But is it correct ?

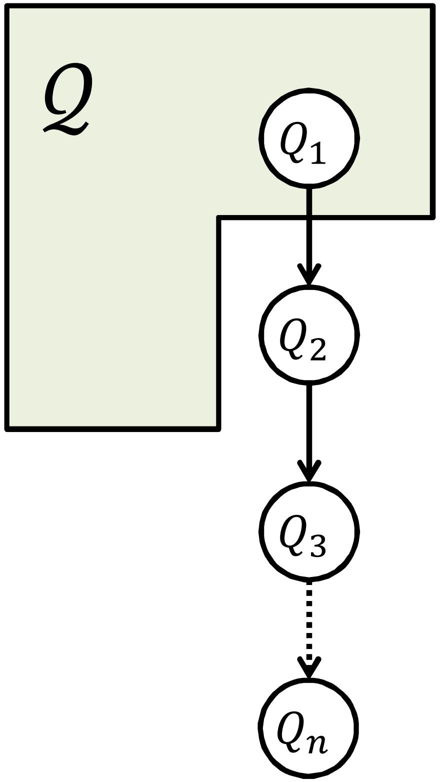
  By removing a non-minimal state we may be preventing the discovery of a minimal state in the future!

Subsumption test

Proposition: Let A be an NFA and let $B = NFAtoDFA(A)$.
After termination of *UnivNFA(A)* the set $\mathcal{Q}$ contains all minimal states of B.

Proof: Assume the contrary. Then B has a shortest path $Q_1 \rightarrow Q_2 \rightarrow \ldots \rightarrow Q_n$ such that

- $Q_1 \in \mathcal{Q}$ (after termination), and
- $Q_n \notin \mathcal{Q}$ and Q_n is minimal.



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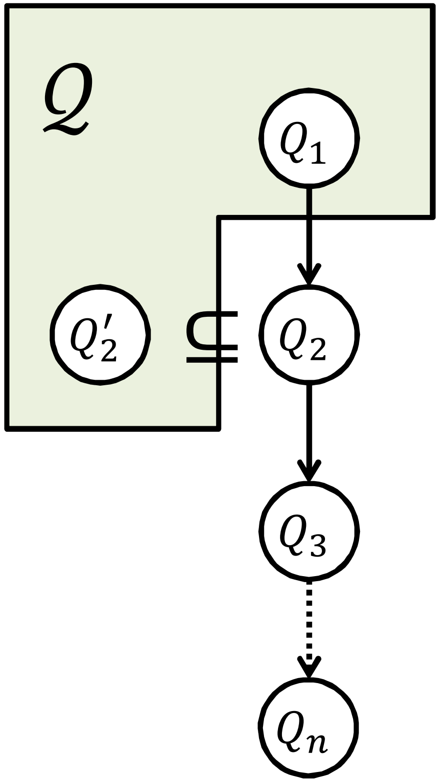

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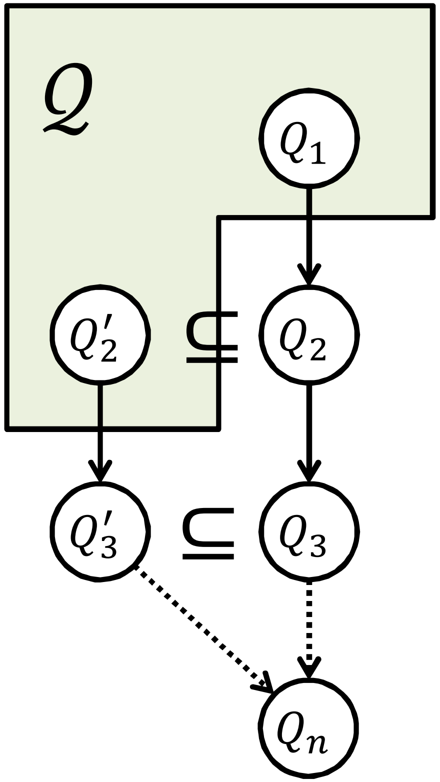

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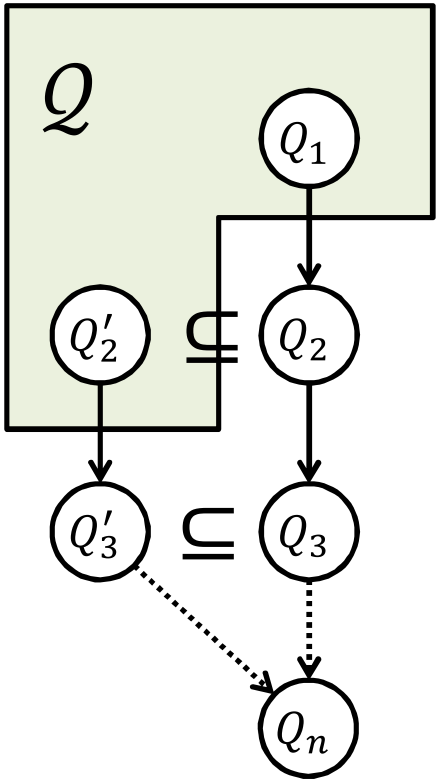
So the path $Q_2' \rightarrow \ \dots \ \rightarrow Q_n'$ satisfies

- $Q_2' \in \mathcal{Q}$ (after termination), and
- Q_n' is minimal.

contradicting that $Q_1 \rightarrow Q_2 \rightarrow \ \dots \ \rightarrow Q_n$ is shortest.



Inclusion

- Proposition: The inclusion problem is PSPACE-complete.
- Proof:

Membership in PSPACE. By Savitch´s theorem it suffices to give a nondeterministic algorithm for non-inclusion. For this, guess letter by letter a word, storing the sets of states Q_1', Q_2' reached by both NFAs on the word guessed so far. Stop when Q_1'contains a final state, but Q_2' does not.

PSPACE-hardness. A is universal iff $L(A) \supseteq L(B)$, where B is the one-state DFA for Σ^*.

Deciding inclusion

- Algorithm: use $L(A_1) \subseteq L(A_2)$ iff $L(A_1) \cap \overline{L(A_2)} = \emptyset$
- Concatenate four algorithms:
  (1) determinize A_2 $\Longrightarrow B_1$
  (2) complement the result $\Longrightarrow B_2$
  (3) intersect B_2 with A_1 $\Longrightarrow B_3$
  (4) check for emptiness of B_3.
- State of B_3: pair (q, Q), with q state of A_1 and Q (sub)set of states of A_2
- Easy optimizations:
  - store only the states of B_3, not its transitions;
  - do not fully construct B_1, then B_2 , then B_3; instead, construct directly the states of B_3;
  - check for emptiness while constructing B_3.

Deciding inclusion

- Further optimization: subsumption test.

Algorithm 18 NFA inclusion check.

InclNFA(A_1, A_2)

Input: NFAs $A_1 = (Q_1, \Sigma, \delta_1, Q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, Q_{02}, F_2)$

Output: true if $\mathcal{L}(A_1) \subseteq \mathcal{L}(A_2)$, **false** otherwise

$Q \leftarrow \emptyset$;
$W \leftarrow \{[q_{01}, Q_{02}] : q_{01} \in Q_{01}\}$
while $W \neq \emptyset$ **do**
  pick $[q_1, Q'_2]$ **from** W
  if $(q_1 \in F_1)$ **and** $(Q'_2 \cap F_2 = \emptyset)$ **then return false**
  add $[q_1, Q'_2]$ **to** Q
  for all $a \in \Sigma$ **do**
    $Q''_2 \leftarrow \bigcup_{q_2 \in Q'_2} \delta_2(q_2, a)$
    for all $q'_1 \in \delta_1(q_1, a)$ **do**
    if $W \cup Q$ contains no $[q''_1, Q'''_2]$ s.t. $q''_1 = q'_1$ and $Q'''_2 \subseteq Q''_2$ **then**
      add $[q'_1, Q''_2]$ **to** W
return true

Deciding inclusion

- Complexity:
  - Let A_1, A_2 be NFAs with n_1, n_2 states over an alphabet with k letters.
  - Without the subsumption test:
    - The while-loop is executed at most $n_1 \cdot 2^{n_2}$ times.
    - The outer for-loop is executed k times.
    - Line 8 takes $O(n_2^2)$ time.
    - The inner for-loop is executed at most n_1 times.
    - Line 19 (without subsumption!) takes constant time.
    - Overall: $O(k \cdot n_1^2 \cdot n_2^2 \cdot 2^{n_2})$ time.
  - With the subsumption case the worst-case complexity is higher. Exercise: give an upper bound.

Deciding inclusion

- Important special case: A_1 is an NFA, A_2 is a DFA.
  - Complementing A_2 is now easy.
  - The while-loop is executed $O(n_1 \cdot n_2)$ times.
  - The outer for-loop is executed k times.
  - Line 8 takes constant time
  - The inner for-loop is executed $O(n_1)$ times
  - Line 10 (without subsumption) takes constant time
  - Overall: $O(k \cdot n_1^2 \cdot n_2)$ time.
- Checking equality: check inclusion in both directions.