Automata and Formal Languages

(Automaten und formale Sprachen)

Winter 2022/23

Syllabus

Course schedule

Lectures

Javier Esparza (esparza@in.tum.de)

Monday:	10:15 – 11:45
	Room: MI HS2 5604.EG.011
Tuesday:	10:45: - 12:15
	Room: MI HS2 5604.EG.011

Lectures are livestreamed (https://live.rbg.tum.de/)

Exercises

A.R. Balasubramanian (ayikudir@in.tum.de)
Marijana Lazic (lazic@in.tum.de)

Thursday:	14:15 - 15:45
	Room: 02.13.010, 5613.02.010

Grading

Written exam

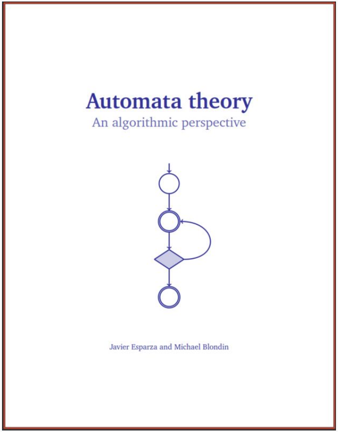
- end of the term
- planned as on-site exam
- 40 points
- retake at the beginning of next term

Exercises not graded!

Points	Grade
[36, 40]	1,0
[34, 36)	1,3
[32, 34)	1,7
[30, 32)	2,0
[28, 30)	2,3
[26, 28)	2,7
[24, 26)	3,0
[22, 24)	3,3
[19, 22)	3,7
[17, 19)	4,0
[11, 17)	4,3
[5, 11)	4,7
[0, 5)	5,0

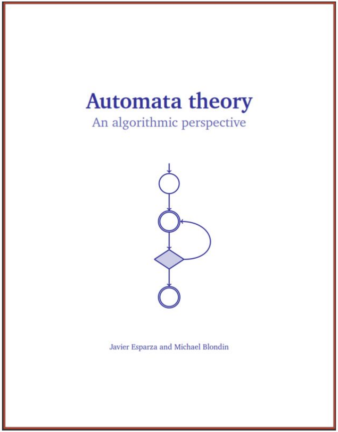
Material

- Lecture notes.
  Over 150 exercises with solutions.
  No need to buy a book.
  Available through moodle
- To be published in 2023 (MIT Press)
  Bug finding grade bonus: 0.3 points.
  Only if you pass the exam.
  Bugs reported through moodle.
  Bonus given depending on quantity and quality of bugs.
  (Effort similar to exercise bonus in bachelor courses.)



Material

- Slides.
- Refresh videos.
- Automata Tutor.



Course content

Automata on finite words

1. Automata classes and conversions
2. Minimization and reduction
3. Boolean operations and tests
4. Operations on relations
5. Operations on finite universes: decision diagrams
6. Automata and logic
7. Pattern-matching, verification, Presburger arithmetic

Course content

Automata on infinite words

8. Automata classes and conversions
9. Boolean operations
10. Emptiness check
11. Verification using temporal logic

Automata theory: Brief recap of basic terminology

Formal languages

An *alphabet* is a nonempty finite set of *letters*

e.g. $\{\mathtt{0}, \mathtt{1}\}$, $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$, $\{[\begin{smallmatrix}0\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\1\end{smallmatrix}]\}$, $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$

A *(finite) word* is a finite sequence of letters

e.g. $\mathtt{1001}$, $\mathtt{hello}$, $[\begin{smallmatrix}1\\0\end{smallmatrix}][\begin{smallmatrix}0\\1\end{smallmatrix}][\begin{smallmatrix}1\\0\end{smallmatrix}]$, $\clubsuit\clubsuit\diamondsuit$, ε

A *language* is a set of words

e.g. $\{\mathtt{1}, \mathtt{10}, \mathtt{100}, \mathtt{1000}, \ldots\}$, $\{\mathtt{aa}, \mathtt{aba}, \mathtt{abbba}, \ldots\}$

Formal languages

An *alphabet* is a nonempty finite set of *letters*

e.g. $\{0, 1\}$, $\{a, b, \ldots, z\}$, $\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\}$, $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$

A *(finite) word* is a finite sequence of letters

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A *language* is a set of words

e.g. $\{1, 10, 100, 1000, \ldots\}$, $\{aa, aba, abbba, \ldots\}$

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e.g. `1001`, `hello`, $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\clubsuit\clubsuit\diamondsuit$, ε

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Formal languages

Let $u = a_1 \cdots a_n$ and $v = b_1 \cdots b_m$ be words

Concatenation:

$$\begin{aligned} u \cdot v &= uv = a_1 \cdots a_n b_1 \cdots b_m \\ \varepsilon \cdot u &= u \;\;= u \cdot \varepsilon \end{aligned}$$

Exponentiation: $u^0 = \varepsilon$, $u^{k+1} = u^k \cdot u$

e.g. $\mathtt{a}^0 = \varepsilon$, $\mathtt{a}^1 = \mathtt{a}$, $(\mathtt{hallo})^2 = \mathtt{hallohallo}$,
$\mathtt{1}^5 = \mathtt{11111}$, $\varepsilon^{1000} = \varepsilon$, $\mathtt{ab} \cdot \mathtt{cde} = \mathtt{abcde}$

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Formal languages

Let L and L' be languages over alphabet Σ

Concatenation: $L \cdot L' = LL' = \{u \cdot v : u \in L, v \in L'\}$

Exponentiation: $L^0 = \{\varepsilon\}$, $L^{k+1} = L^k \cdot L$

Iteration: $L^* = L^0 \cup L^1 \cup L^2 \cup \cdots$

Complement: $\overline{L} = \Sigma^* \setminus L$

e.g. $$\begin{aligned} \{\mathrm{aa}, \mathrm{bb}\} \cdot \{\mathrm{c}, \mathrm{d}\} &= \{\mathrm{aac}, \mathrm{aad}, \mathrm{bbc}, \mathrm{bbd}\}, \\ \{\mathrm{aa}, \mathrm{b}\}^* &= \{\varepsilon, \mathrm{aa}, \mathrm{b}, \mathrm{aaaa}, \mathrm{aab}, \ldots\} \end{aligned}$$

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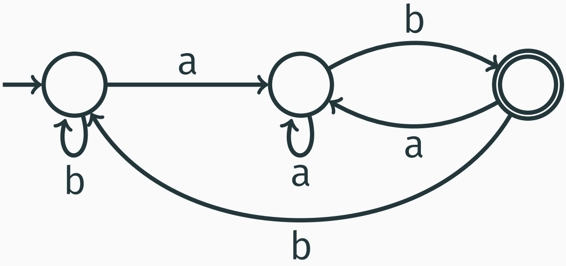
Deterministic finite automata (DFA)

- *States*: nonempty finite set Q
- *Alphabet*: nonempty finite set Σ
- *Transitions*: $\delta : Q \times \Sigma \to Q$
- *Initial state*: $q_0 \in Q$
- *Final states*: $F \subseteq Q$



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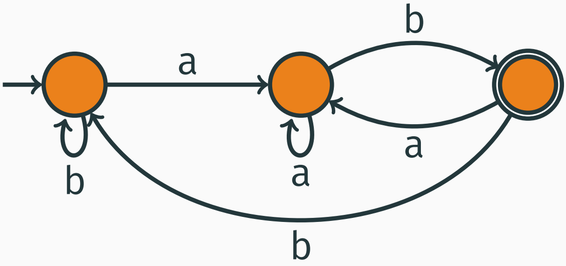

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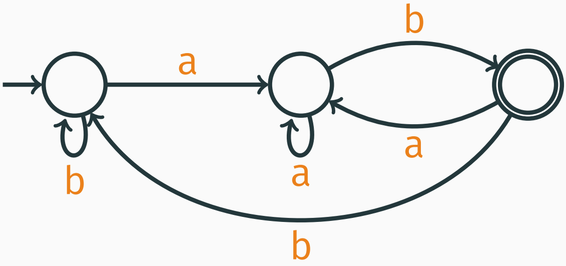

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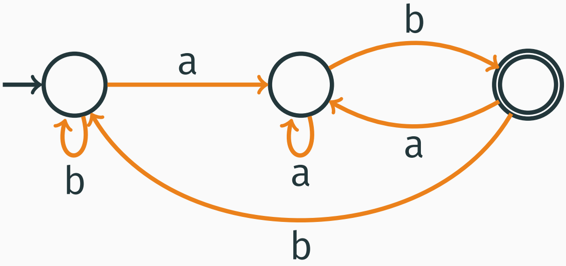

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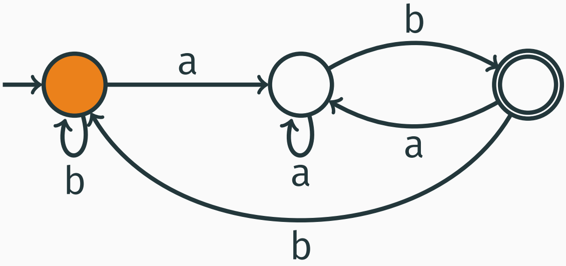

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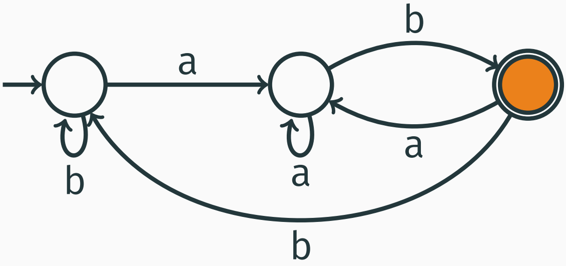
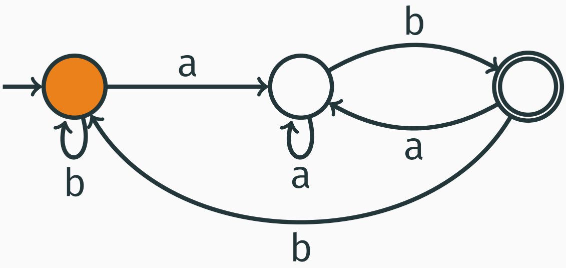


w = aabab



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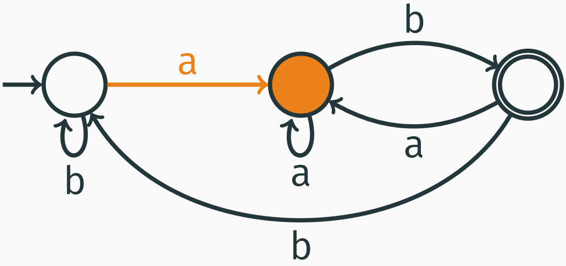

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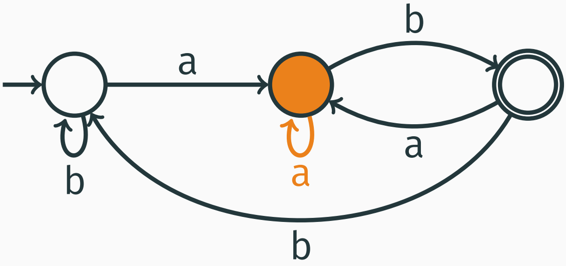

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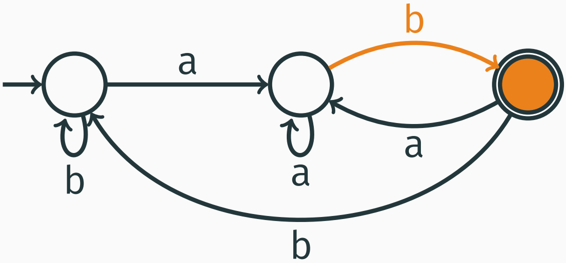

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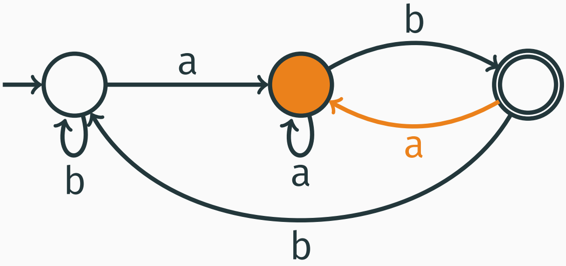

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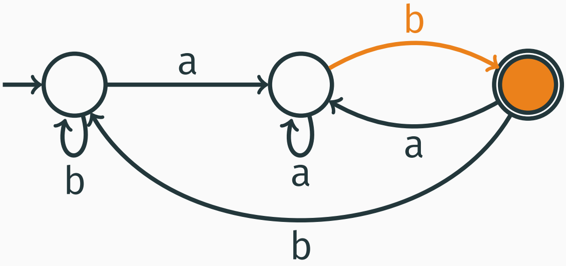
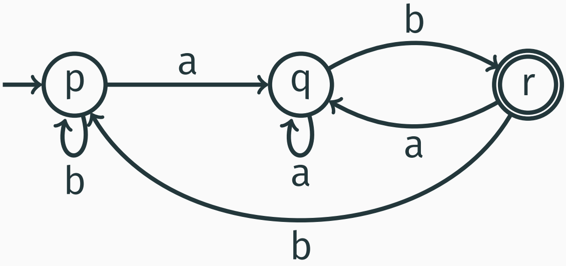


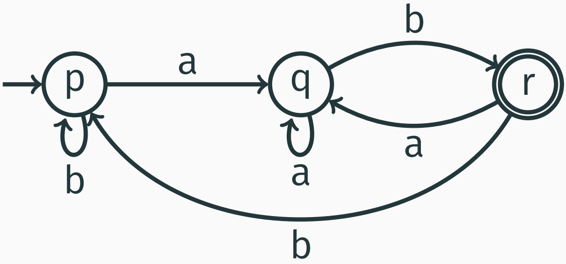
Deterministic finite automata (DFA)

$$p \xrightarrow{a} q \xrightarrow{a} q \xrightarrow{b} r \xrightarrow{a} q \xrightarrow{b} r$$



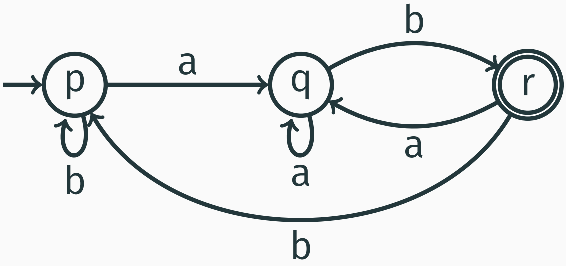
Deterministic finite automata (DFA)

$$p \xrightarrow{\text{aabab}} r$$



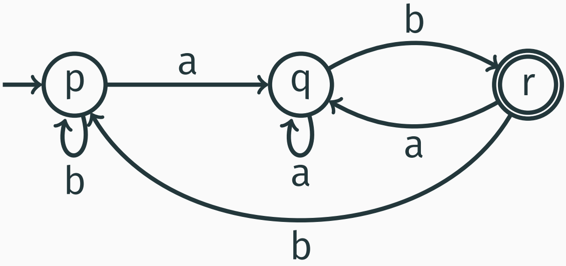
Deterministic finite automata (DFA)

$$L(A) = \{w \in \Sigma^* : \exists q \in F \text{ s.t. } q_0 \xrightarrow{w} q\}$$



Deterministic finite automata (DFA)

$$L(A) = \{w \in \Sigma^* : \quad w \text{ ends with ab} \quad \}$$



Nondeterministic finite automata (NFA)

- *States*: nonempty finite set Q
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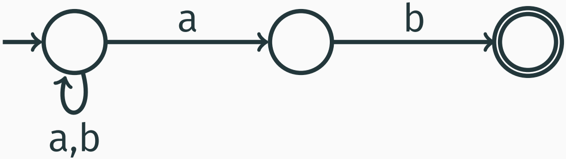

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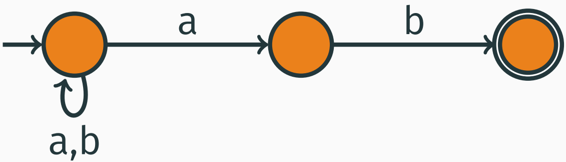

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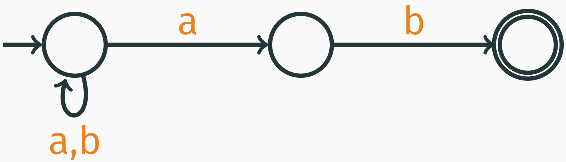

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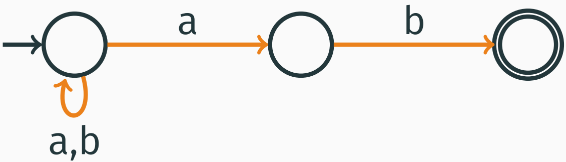

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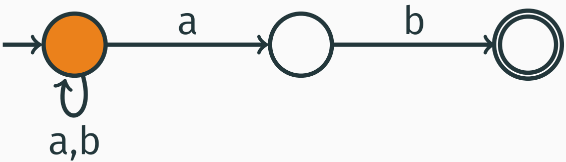

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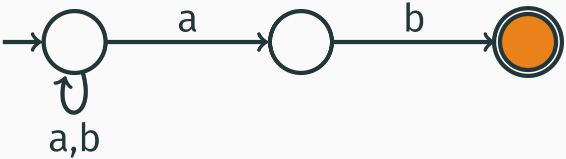

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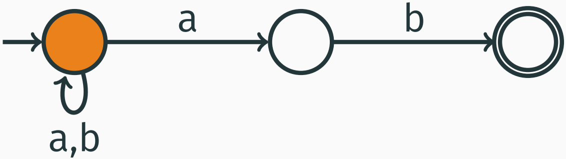

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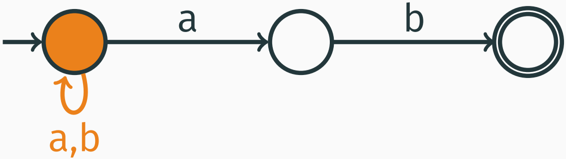

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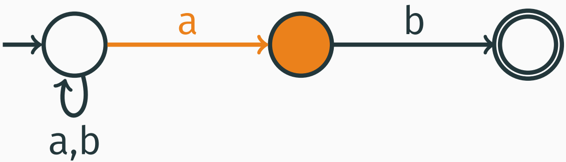

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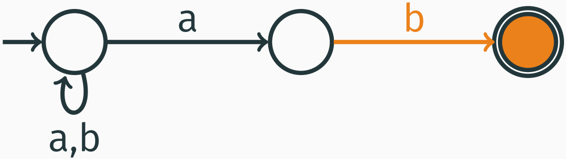

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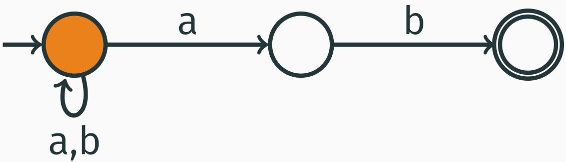

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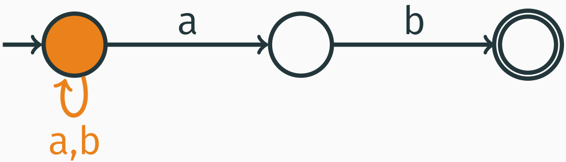

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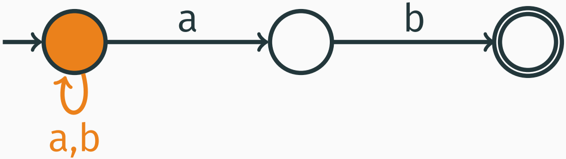

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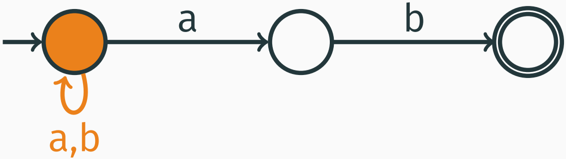
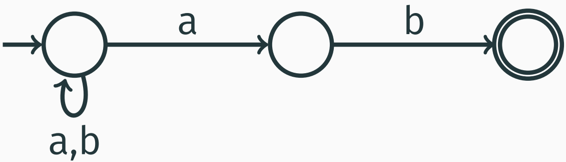


Nondeterministic finite automata (NFA)

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} p_n$$

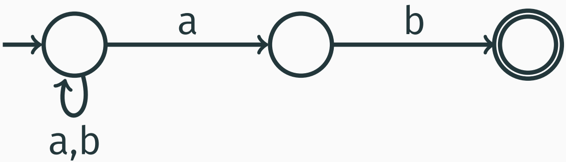
$$\Updownarrow$$

$$p_i \in \delta(p_{i-1}, a_i) \text{ for every } 0 < i \leq n$$



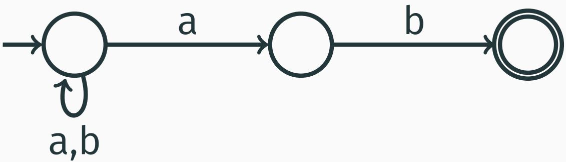
Nondeterministic finite automata (NFA)

$$L(A) = \{w \in \Sigma^* : \exists q_0 \in Q_0, q \in F \text{ s.t. } q_0 \xrightarrow{w} q\}$$



Nondeterministic finite automata (NFA)

$$L(A) = \{w \in \Sigma^* : \quad w \text{ ends with ab} \quad \}$$



Regular expressions

$$r ::= \emptyset \mid \varepsilon \mid a \mid r_1 r_2 \mid r_1 + r_2 \mid r^*$$

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$$\begin{aligned} L(\emptyset) &= \emptyset & \quad L(r_1 r_2) &= L(r_1) \cdot L(r_2) \\ L(\varepsilon) &= \{\varepsilon\} & \quad L(r_1 + r_2) &= L(r_1) \cup L(r_2) \\ L(a) &= \{a\} & \quad L(r^*) &= L(r)^* \end{aligned}$$

Regular expressions

$$L(\,(a+b)^{*}ab\,) = \{w \in \{a,b\}^{*} : w \text{ ends with } \mathtt{ab}\}$$

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More examples

$$L = \{w \in \{\mathrm{a}, \mathrm{b}\}^* : w \text{ contains } \mathrm{aaa}\}$$

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Regular expression?

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Regular expression?

$$(\mathrm{a} + \mathrm{b})^* \mathrm{aaa} (\mathrm{a} + \mathrm{b})^*$$

More examples

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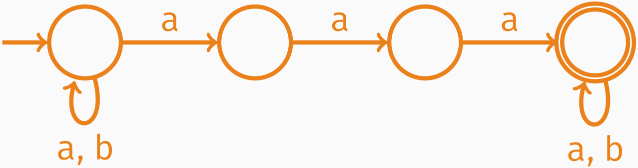

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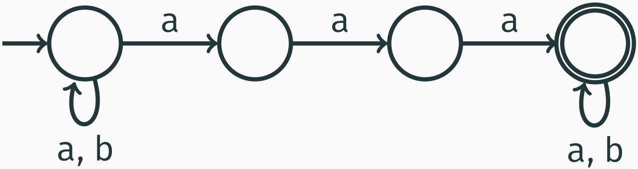
DFA?



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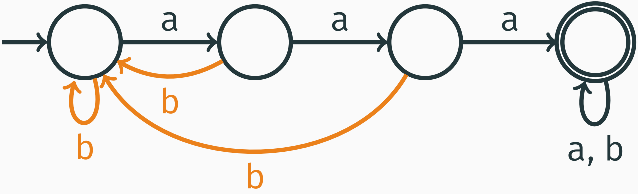

More examples

$$L = \{w \in \{0,1\}^* : w \text{ contains } \begin{array}{l}\text{an even number of 0 or} \\ \text{an odd number of 1}\end{array} \}$$

More examples

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Regular expression?

More examples

$$L = \{w \in \{0,1\}^* : w \text{ contains an even number of } 0 \text{ or an odd number of } 1\}$$

Regular expression?

$$(1^*01^*0)^*1^* + (0^*10^*1)^*0^*10^*$$

More examples

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NFA?

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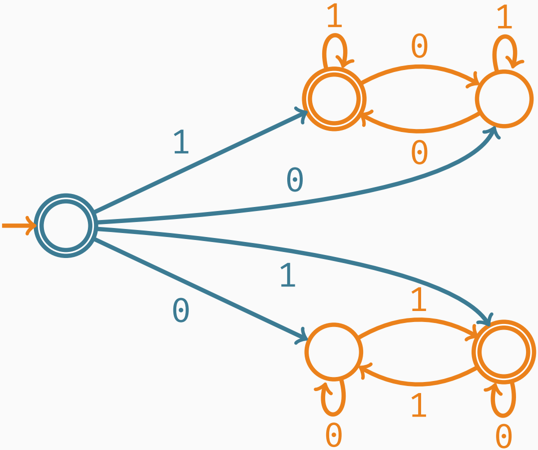
NFA?



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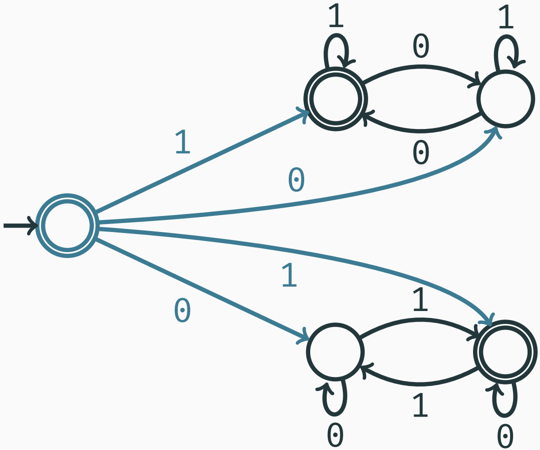

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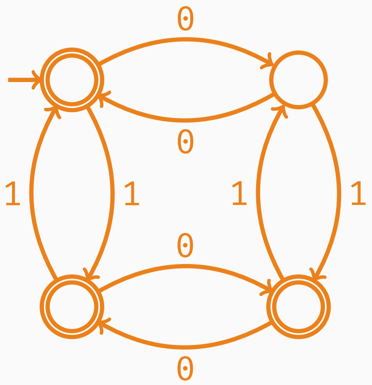
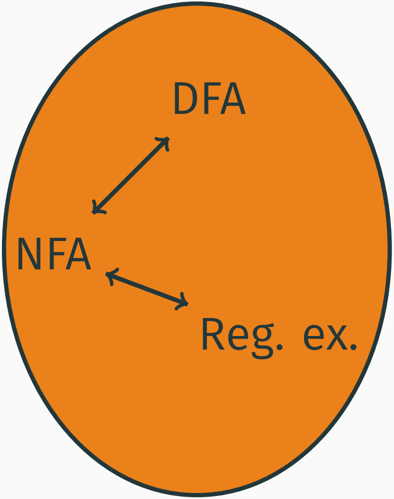
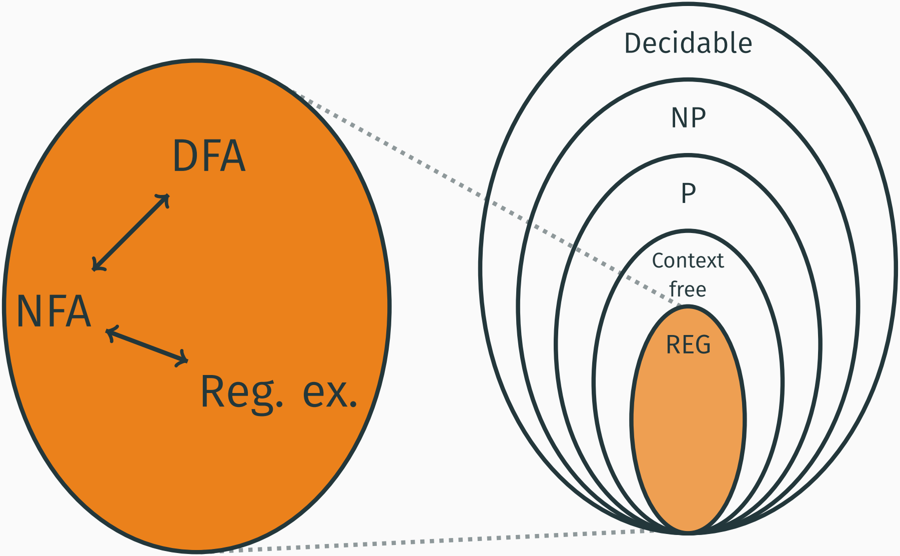



DFA
NFA
Reg. ex.

Regular languages



Regular languages