Automata and Formal Languages — Exercise Sheet 13

Exercise 13.1
Let AP $= \{p, q\}$ and let $\Sigma = 2^{\text{AP}}$. Give LTL formulas for the following ω-languages:

(a) $\{p, q\}\,\emptyset\,\Sigma^\omega$

(b) $\Sigma^*\,\{q\}^\omega$

(c) $\Sigma^*\,(\{p\} + \{p, q\})\,\Sigma^*\,\{q\}\,\Sigma^\omega$

(d) $\{p\}^*\,\{q\}^*\,\emptyset^\omega$

In (a) and (d) the $\emptyset$ symbol stands for the letter $\emptyset \in \Sigma$, and not for the empty ω-language.

Exercise 13.2
Let AP $= \{p, q\}$ and let $\Sigma = 2^{\text{AP}}$. Give Büchi automata for the ω-languages over Σ defined by the following LTL formulas:

(a) $\mathbf{XG}\neg p$

(b) $(\mathbf{GF}p) \rightarrow (\mathbf{F}q)$

(c) $p \wedge \neg(\mathbf{XF}p)$

(d) $\mathbf{G}(p\ \mathbf{U}\ (p \rightarrow q))$

(e) $\mathbf{F}q \rightarrow (\neg q\ \mathbf{U}\ (\neg q \wedge p))$

Exercise 13.3
Let $\mathcal{V} \in \{\mathbf{F}, \mathbf{G}\}^*$ be a sequence made of the temporal operators $\mathbf{F}$ and $\mathbf{G}$. Show that $\mathbf{FG}p \equiv \mathcal{V}\mathbf{FG}p$ and $\mathbf{GF}p \equiv \mathcal{V}\mathbf{GF}p$.

Exercise 13.4
Say which of the following equivalences hold. For every equivalence that does not hold give an instantiation of φ and ψ together with a computation that disproves the equivalence.

(a) $\mathbf{F}(\varphi \vee \psi) \equiv \mathbf{F}\varphi \vee \mathbf{F}\psi$

(b) $\mathbf{F}(\varphi \wedge \psi) \equiv \mathbf{F}\varphi \wedge \mathbf{F}\psi$

(c) $\mathbf{G}(\varphi \vee \psi) \equiv \mathbf{G}\varphi \vee \mathbf{G}\psi$

(d) $(\varphi \vee \psi)\ \mathbf{U}\ \rho \equiv (\varphi\ \mathbf{U}\ \rho) \vee (\psi\ \mathbf{U}\ \rho)$

(e) $\mathbf{GF}(\varphi \wedge \psi) \equiv \mathbf{GF}\varphi \wedge \mathbf{GF}\psi$

(f) $\mathbf{X}(\varphi\ \mathbf{U}\ \psi) \equiv (\mathbf{X}\varphi\ \mathbf{U}\ \mathbf{X}\psi)$

Solution 13.1

(a) $(p \wedge q) \wedge \mathbf{X}(\neg p \wedge \neg q)$

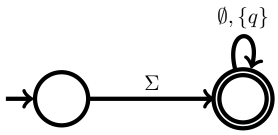
(b) $\mathbf{FG}(\neg p \wedge q)$

(c) $\mathbf{F}(p \wedge \mathbf{XF}(\neg p \wedge q))$

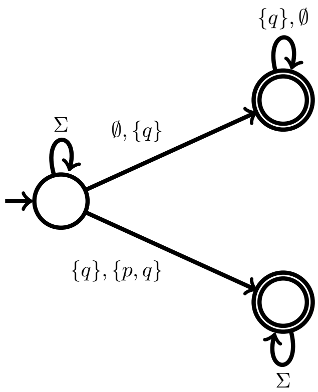
(d) $(p \wedge \neg q)\ \mathbf{U}\ ((\neg p \wedge q)\ \mathbf{U}\ \mathbf{G}(\neg p \wedge \neg q))$

Solution 13.2

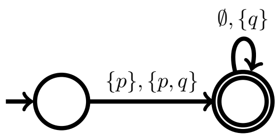
(a)



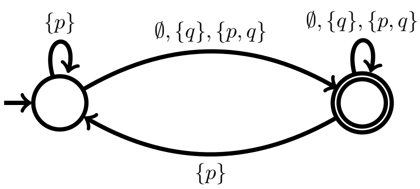
(b) Note that $(\mathbf{GF}p) \rightarrow (\mathbf{F}q) \equiv \neg(\mathbf{GF}p) \vee (\mathbf{F}q) \equiv (\mathbf{FG}\neg p) \vee (\mathbf{F}q)$. We construct Büchi automata for $\mathbf{FG}\neg p$ and $\mathbf{F}q$, and take their union:



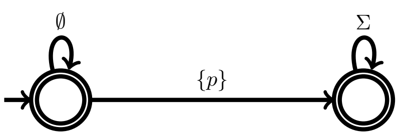
(c) Note that $p \wedge \neg(\mathbf{XF}p) \equiv p \wedge \mathbf{XG}\neg p$. We construct a Büchi automaton for $p \wedge \mathbf{XG}\neg p$:



(d)



(e) Note that $\mathbf{F}q \rightarrow (\neg q\ \mathbf{U}\ (\neg q \wedge p)) \equiv \mathbf{G}\neg q \vee (\neg q\ \mathbf{U}\ (\neg q \wedge p))$. Consider this case split over the occurence of a p : computations that satisfy the formula either have no occurrence of p, in which case they must satisfy the first part of the $\vee$ (i.e. $\mathbf{G}\neg q$), or they have a first occurrence of p with no q before or at the same time:



Solution 13.3

Given LTL formulas φ and ψ, we denote by $\varphi \models \psi$ that every computation satisfying φ satisfies ψ. Note that $\varphi \equiv \sigma$ iff $\varphi \models \psi$ and $\psi \models \varphi$. It is readily seen that the following holds:

$$\begin{aligned} \mathbf{FF}\varphi &\equiv \mathbf{F}\varphi, && (1) \\ \mathbf{GG}\varphi &\equiv \mathbf{G}\varphi, && (2) \\ \mathbf{G}\varphi \models \varphi \text{ and } \varphi &\models \mathbf{F}\varphi. && (3) \end{aligned}$$

Let us show that (a) $\mathbf{FG}\varphi \equiv \mathbf{GFG}\varphi$ and (b) $\mathbf{GF}\varphi \equiv \mathbf{FGF}\varphi$.

(a) We have $\mathbf{GFG}\varphi \models \mathbf{FG}\varphi$ by (3). Let $\sigma \models \mathbf{FG}\varphi$. There exists $i \geq 0$ such that $\sigma^j \models \varphi$ for every $j \geq i$. Thus, for every $k \geq 0$ there is some $\ell \geq 0$ such that $(\sigma^k)^\ell \models \mathbf{G}\varphi$. Indeed, if $k \geq i$ then take $\ell = 0$, and if $k < i$ then take $\ell = i - k$. Therefore, we have $\sigma^k \models \mathbf{FG}\varphi$ for every $k \geq 0$, and hence $\sigma \models \mathbf{GFG}\varphi$. This means that $\mathbf{FG}\varphi \models \mathbf{GFG}\varphi$.

(b) We have $\mathbf{GF}\varphi \models \mathbf{FGF}\varphi$ by (3). It is the case that $\mathbf{FGF}\varphi \models \mathbf{GF}\varphi$. Indeed, if there exists $i \geq 0$ such that $\sigma^j \models \varphi$ holds for infinitely many $j \geq i$, then, in particular, $\sigma^j \models \varphi$ holds for infinitely many $j \geq 0$.

We prove $\mathbf{FG}\varphi \equiv \mathcal{V}\mathbf{FG}\varphi$ by induction on the length of $\mathcal{V}$. If $\mathcal{V} = \varepsilon$, then we are done. If $\mathcal{V} = \mathcal{U}\mathbf{F}$, then we have $\mathcal{V}\mathbf{FG}\varphi \equiv \mathcal{U}\mathbf{FG}\varphi$ by (1). If $\mathcal{V} = \mathcal{U}\mathbf{G}$, then we have the same equivalence by (a). By induction hypothesis we get $\mathcal{U}\mathbf{FG}\varphi \equiv \mathbf{FG}\varphi$. The other equivalence is proved similarly using (2) and (b). $\square$

Solution 13.4

(a) True, since:

$$\begin{aligned} \sigma \models \mathbf{F}(\varphi \lor \psi) &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \lor \psi) \\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi) \lor (\sigma^k \models \psi) \\ &\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi) \lor (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi) \\ &\iff \sigma \models \mathbf{F}\varphi \lor \mathbf{F}\psi. \end{aligned}$$

(b) False. Let $\sigma = \{p\}\{q\}\emptyset^\omega$. We have $\sigma \models \mathbf{F}p \land \mathbf{F}q$ and $\sigma \not\models \mathbf{F}(\varphi \land \psi)$.

(c) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \models \mathbf{G}(p \lor q)$ and $\sigma \not\models \mathbf{G}p \lor \mathbf{G}q$.

(d) False. Let $\sigma = \{p\}\{q\}\{r\}\emptyset^\omega$. We have $\sigma \models (p \lor q)\ \mathbf{U}\ r$ and $\sigma \not\models (p\ \mathbf{U}\ r) \lor (q\ \mathbf{U}\ r)$.

(e) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \not\models \mathbf{GF}(p \land q)$ and $\sigma \models \mathbf{GF}p \land \mathbf{GF}q$.

(f) True, since:

$$\begin{aligned} \sigma \models \mathbf{X}(\varphi\ \mathbf{U}\ \psi) &\iff \sigma^1 \models (\varphi\ \mathbf{U}\ \psi) \\ &\iff \exists k \geq 0 : (\sigma^1)^k \models \varphi \text{ and } \forall 0 \leq i < k\ (\sigma^1)^i \models \psi \\ &\iff \exists k \geq 0 : (\sigma^k)^1 \models \varphi \text{ and } \forall 0 \leq i < k\ (\sigma^i)^1 \models \psi \\ &\iff \exists k \geq 0 : \sigma^k \models \mathbf{X}\varphi \text{ and } \forall 0 \leq i < k\ (\sigma^i \models \mathbf{X}\psi) \\ &\iff \sigma \models (\mathbf{X}\varphi)\ \mathbf{U}\ (\mathbf{X}\psi). \end{aligned}$$