Automata and Formal Languages — Exercise Sheet 13

Exercise 13.1

Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. Give Büchi automata for the ω-languages over Σ defined by the following LTL formulas:

(a) $\mathbf{XG}\neg p$

(b) $(\mathbf{GF}p) \rightarrow (\mathbf{F}q)$

(c) $p \wedge \neg(\mathbf{XF}p)$

(d) $\mathbf{G}(p\, \mathbf{U}\, (p \rightarrow q))$

(e) $\mathbf{F}q \rightarrow (\neg q\, \mathbf{U}\, (\neg q \wedge p))$

Exercise 13.2

Let $A = (Q, \Sigma, \delta, q_0, F)$ be an automaton such that $Q = P \times [n]$ for some finite set P and $n \geq 1$. Automaton A models a system made of n processes. A state $(p, i) \in Q$ represents the current global state p of the system, and the last process i that was executed.

We define two predicates exec_j and enab_j over Q indicating whether process j is respectively executed and enabled. More formally, for every $q = (p, i) \in Q$ and $j \in [n]$, let

$$\begin{aligned} \mathrm{exec}_j(q) &\iff i = j, \\ \mathrm{enab}_j(q) &\iff (p, i) \rightarrow (p', j) \text{ for some } p' \in P. \end{aligned}$$

(a) Give LTL formulas over Q^ω for the following statements:

  (i) All processes are executed infinitely often.

  (ii) If a process is enabled infinitely often, then it is executed infinitely often.

  (iii) If a process is eventually permanently enabled, then it is executed infinitely often.

(b) The three above properties are known respectively as *unconditional*, *strong* and *weak* fairness. Show the following implications, and show that the reverse implications do not hold:

$$\text{unconditional fairness} \implies \text{strong fairness} \implies \text{weak fairness}.$$

Exercise 13.3
Prove or disprove:

(a) $\mathbf{F}(\varphi \vee \psi) \equiv \mathbf{F}\varphi \vee \mathbf{F}\psi$

(b) $\mathbf{F}(\varphi \wedge \psi) \equiv \mathbf{F}\varphi \wedge \mathbf{F}\psi$

(c) $\mathbf{G}(\varphi \vee \psi) \equiv \mathbf{G}\varphi \vee \mathbf{G}\psi$

(d) $\mathbf{G}(\varphi \wedge \psi) \equiv \mathbf{G}\varphi \wedge \mathbf{G}\psi$

(e) $(\varphi \vee \psi)\ \mathbf{U}\ \rho \equiv (\varphi\ \mathbf{U}\ \rho) \vee (\psi\ \mathbf{U}\ \rho)$

(f) $\rho\ \mathbf{U}\ (\varphi \vee \psi) \equiv (\rho\ \mathbf{U}\ \varphi) \vee (\rho\ \mathbf{U}\ \psi)$

Exercise 13.4
Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. An LTL formula is a tautology if it is satisfied by all computations. Which of the following LTL formulas are tautologies?

(a) $\mathbf{G}p \rightarrow \mathbf{F}p$

(b) $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$

(c) $\mathbf{F}\mathbf{G}p \vee \mathbf{F}\mathbf{G}\neg p$

(d) $\neg\mathbf{F}p \rightarrow \mathbf{F}\neg\mathbf{F}p$

(e) $(\mathbf{G}p \rightarrow \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \vee q))$

(f) $\neg(p\ \mathbf{U}\ q) \leftrightarrow (\neg p\ \mathbf{U}\ \neg q)$

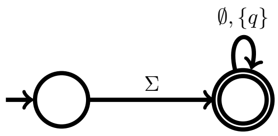
(g) $\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p)$

(h) $(\mathbf{G}\mathbf{F}p \wedge \mathbf{G}\mathbf{F}q) \rightarrow \mathbf{G}(p\ \mathbf{U}\ q)$

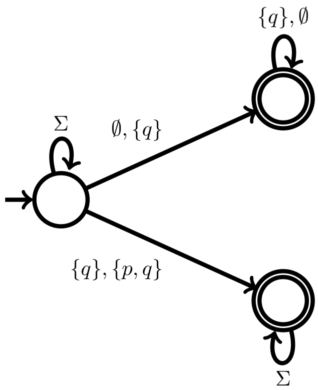
(i) $\mathbf{G}(p\ \mathbf{U}\ q) \rightarrow (\mathbf{G}\mathbf{F}p \vee \mathbf{G}\mathbf{F}q)$

Solution 13.1

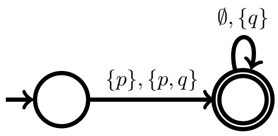
(a)



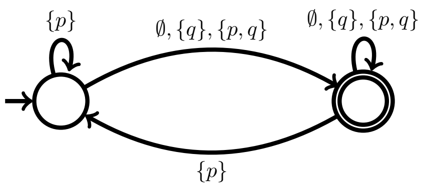
(b) Note that $(\mathbf{GF}p) \to (\mathbf{F}q) \equiv \neg(\mathbf{GF}p) \vee (\mathbf{F}q) \equiv (\mathbf{FG}\neg p) \vee (\mathbf{F}q)$. We construct Büchi automata for $\mathbf{FG}\neg p$ and $\mathbf{F}q$, and take their union:



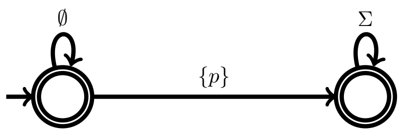
(c) Note that $p \wedge \neg(\mathbf{XF}p) \equiv p \wedge \mathbf{XG}\neg p$. We construct a Büchi automaton for $p \wedge \mathbf{XG}\neg p$:



(d)



(e)



Solution 13.2

(a) (i) $\bigwedge_{j \in [n]} \mathbf{GF}\ \text{exec}_j$

(ii) $\bigwedge_{j \in [n]} (\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j)$

(iii) $\bigwedge_{j \in [n]} (\mathbf{FG}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j)$

(b) • Unconditional fairness implies strong fairness. For the sake of contradiction, suppose unconditional fairness holds for some execution σ, but not strong fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j)$. Thus,

$$\begin{aligned}
\sigma &\not\models (\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) && \iff \\
\sigma &\models \neg(\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) && \iff \\
\sigma &\models \neg(\neg\mathbf{GF}\ \text{enab}_j \vee \mathbf{GF}\ \text{exec}_j) && \iff \\
\sigma &\models \mathbf{GF}\ \text{enab}_j \wedge \neg\mathbf{GF}\ \text{exec}_j && \implies \\
\sigma &\models \neg\mathbf{GF}\ \text{exec}_j
\end{aligned}$$

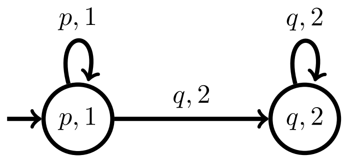
which contradicts unconditional fairness. □

- Strong fairness implies weak fairness. For the sake of contradiction, suppose strong fairness holds for some execution σ, but not weak fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\mathbf{FG}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j)$. Thus,

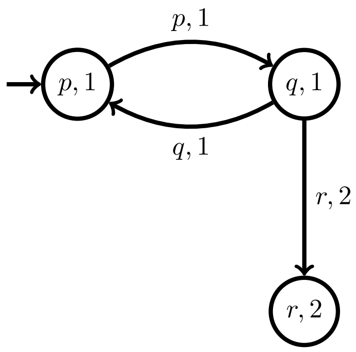
$$\begin{aligned}
\sigma &\not\models (\mathbf{FG}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) && \iff \\
\sigma &\models \neg(\mathbf{FG}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) && \iff \\
\sigma &\models \neg(\neg\mathbf{FG}\ \text{enab}_j \vee \mathbf{GF}\ \text{exec}_j) && \iff \\
\sigma &\models \mathbf{FG}\ \text{enab}_j \wedge \neg\mathbf{GF}\ \text{exec}_j && \implies \\
\sigma &\models \mathbf{GF}\ \text{enab}_j \wedge \neg\mathbf{GF}\ \text{exec}_j && \iff \\
\sigma &\models \neg(\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) && \iff \\
\sigma &\not\models \mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j
\end{aligned}$$

which contradicts strong fairness. □

- Strong fairness does not imply unconditional fairness. Execution $(p,1)(q,2)^\omega$ of the automaton below satisfies strong fairness, but not unconditional fairness.



- Weak fairness does not imply strong fairness. Execution $((p,1)(q,1))^\omega$ of the automaton below satisfies weak fairness, but not strong fairness.



Solution 13.3

(a) True, since:

$$\begin{aligned}
\sigma \models \mathbf{F}(\varphi \vee \psi) &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \vee \psi) \\
&\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi) \vee (\sigma^k \models \psi) \\
&\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi) \vee (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi) \\
&\iff \sigma \models \mathbf{F}\varphi \vee \mathbf{F}\psi.
\end{aligned}$$

(b) False. Let $\sigma = \{p\}\{q\}\emptyset^\omega$. We have $\sigma \models \mathbf{F}p \wedge \mathbf{F}q$ and $\sigma \not\models \mathbf{F}(\varphi \wedge \psi)$.

(c) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \models \mathbf{G}(p \vee q)$ and $\sigma \not\models \mathbf{G}p \vee \mathbf{G}q$.

(d) True, since:

$$\begin{aligned}\sigma \models \mathbf{G}(\varphi \land \psi) &\iff \forall k \geq 0\ \sigma^k \models (\varphi \land \psi)\\ &\iff \forall k \geq 0\ (\sigma^k \models \varphi) \land (\sigma^k \models \psi)\\ &\iff (\forall k \geq 0\ \sigma^k \models \varphi) \land (\forall k \geq 0\ \sigma^k \models \psi)\\ &\iff \sigma \models \mathbf{G}\varphi \land \mathbf{G}\psi.\end{aligned}$$

(e) False. Let $\sigma = \{p\}\{q\}\{r\}\emptyset^\omega$. We have $\sigma \models (p \lor q)\ \mathbf{U}\ r$ and $\sigma \not\models (p\ \mathbf{U}\ r) \lor (q\ \mathbf{U}\ r)$.

(f) True, since:

$$\begin{aligned}\sigma \models \rho\ \mathbf{U}\ (\varphi \lor \psi) &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \lor \psi) \land \forall 0 \leq i < k\ \sigma^i \models \rho\\ &\iff \exists k \geq 0 \text{ s.t. } ((\sigma^k \models \varphi) \lor (\sigma^k \models \psi)) \land \forall 0 \leq i < k\ \sigma^i \models \rho\\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi \land \forall 0 \leq i < k\ \sigma^i \models \rho) \lor (\sigma^k \models \psi \land \forall 0 \leq i < k\ \sigma^i \models \rho)\\ &\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi \land \forall 0 \leq i < k\ \sigma^i \models \rho) \lor (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi \land \forall 0 \leq i < k\ \sigma^i \models \rho)\\ &\iff \sigma \models (\rho\ \mathbf{U}\ \varphi) \lor (\rho\ \mathbf{U}\ \psi).\end{aligned}$$

□

Solution 13.4

(a) $\mathbf{G}p \to \mathbf{F}p$ is a tautology since

$$\begin{aligned}\sigma \models \mathbf{G}p &\iff \forall k \geq 0\ \sigma^k \models p\\ &\implies \exists k \geq 0\ \sigma^k \models p\\ &\iff \sigma \models \mathbf{F}p.\end{aligned}$$

(b) $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ is a tautology. For the sake of contradiction, suppose this is not the case. There exists σ such that

$$\sigma \models \mathbf{G}(p \to q), \text{ and} \tag{1}$$
$$\sigma \not\models (\mathbf{G}p \to \mathbf{G}q). \tag{2}$$

By (2), we have

$$\begin{aligned}&\sigma \models \mathbf{G}p, \text{ and}\\ &\sigma \not\models \mathbf{G}q.\end{aligned}$$

Therefore, there exists $k \geq 0$ such that $p \in \sigma(k)$ and $q \notin \sigma(k)$ which contradicts (1).

(c) $\mathbf{FG}p \lor \mathbf{FG}\neg p$ is not a tautology since it is not satisfied by $(\{p\}\{q\})^\omega$.

(d) $\neg\mathbf{F}p \to \mathbf{F}\neg\mathbf{F}p$ is a tautology since $\varphi \to \mathbf{F}\varphi$ is a tautology for every formula φ.

(e) $(\mathbf{G}p \to \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \lor q))$ is a tautology. We have

$$\begin{aligned}\mathbf{G}p \to \mathbf{F}q &\equiv \neg\mathbf{G}p \lor \mathbf{F}q && \text{(by def. of implication)}\\ &\equiv \mathbf{F}\neg p \lor \mathbf{F}q\\ &\equiv \mathbf{F}(\neg p \lor q)\\ &\equiv \mathbf{F}(p \to q) && \text{(by def. of implication)}\end{aligned}$$

Therefore, we have to show that

$$\mathbf{F}(p \to q) \leftrightarrow (p\ \mathbf{U}\ (p \to q)).$$

$\leftarrow$) Let σ be such that $\sigma \models (p\ \mathbf{U}\ (p \to q))$. In particular, there exists $k \geq 0$ such that $\sigma^k \models (p \to q)$. Therefore, $\sigma \models \mathbf{F}(p \to q)$.

$\to$) Let σ be such that $\sigma \models \mathbf{F}(p \to q)$. Let $k \geq 0$ be the smallest position such that $\sigma^k \models (p \to q)$. For every $0 \leq i < k$, we have $\sigma^i \not\models (p \to q)$ which is equivalent to $\sigma^i \models p \land \neg q$. Therefore, for every $0 \leq i < k$, we have $\sigma^i \models p$. This implies that $\sigma \models p\ \mathbf{U}\ (p \to q)$.

(f) $\neg(p \, \mathbf{U} \, q) \leftrightarrow (\neg p \, \mathbf{U} \, \neg q)$ is not a tautology. Let $\sigma = \{p\}\{q\}^\omega$. We have $\sigma \not\models \neg(p \, \mathbf{U} \, q)$ and $\sigma \models (\neg p \, \mathbf{U} \, \neg q)$.

(g) $\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p)$ is a tautology since

$$\begin{aligned}
\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p) &\equiv \neg\mathbf{G}(\neg p \vee \mathbf{X}p) \vee (\neg p \vee \mathbf{G}p) && \text{(by def. of implication)} \\
&\equiv \mathbf{F}(p \wedge \neg\mathbf{X}p) \vee \neg p \vee \mathbf{G}p \\
&\equiv \neg\mathbf{G}p \to (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) && \text{(by def. of implication)} \\
&\equiv \mathbf{F}\neg p \to (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) \\
&\equiv \mathbf{F}\neg p \to \mathbf{F}\neg p.
\end{aligned}$$

(h) $(\mathbf{GF}p \wedge \mathbf{GF}q) \Rightarrow \mathbf{G}(p \, \mathbf{U} \, q)$ is not a tautology. Here are two counterexamples: $(\{p\}\,\emptyset\,\{q\})^\omega$ and $\emptyset\,\{p, q\}^\omega$.

(i) $\mathbf{G}(p \, \mathbf{U} \, q) \Rightarrow (\mathbf{GF}p \vee \mathbf{GF}q)$ is a tautology. We prove this by contradiction.

Suppose the formula is not a tautology. Then there exists an execution σ that does not satisfy it, that is, the following holds:

$$\sigma \not\models \mathbf{G}(p \, \mathbf{U} \, q) \Rightarrow (\mathbf{GF}p \vee \mathbf{GF}q).$$

Therefore, we have the following:

$$\sigma \models \mathbf{G}(p \, \mathbf{U} \, q), \tag{3}$$

$$\sigma \not\models \mathbf{GF}p \vee \mathbf{GF}q. \tag{4}$$

First, note that from (3) we know the following:

$$\sigma^k \models p \, \mathbf{U} \, q, \qquad \text{for every } k \geq 0. \tag{5}$$

Second, note that (4) is equivalent to $\sigma \models \mathbf{FG}\neg p \wedge \mathbf{FG}\neg q$, that is $\sigma \models \mathbf{FG}\neg p$ and $\sigma \models \mathbf{FG}\neg q$. Since we have that $\sigma \models \mathbf{FG}\neg q$, by definition of the operator $\mathbf{F}$ we know the following:

$$\sigma^i \models \mathbf{G}\neg q, \qquad \text{for some } i \geq 0. \tag{6}$$

By definition of $\mathbf{G}$ this means the following:

$$\sigma^j \models \neg q, \qquad \text{for every } j \geq i. \tag{7}$$

Let us now focus again on (5) and on the index i defined in (6). From (5) we know that also for this particular index i it holds that $\sigma^i \models p \, \mathbf{U} \, q$. Therefore, by definition of $\mathbf{U}$, we know that there exists an index $l \geq i$ with $\sigma^l \models q$. This contradicts (7), and hence our assumption that the formula is not a tautology is wrong. This shows that the formula is indeed a tautology.