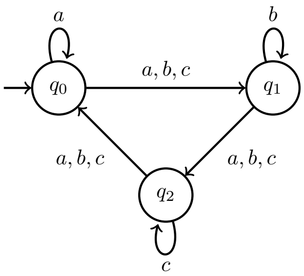
Automata and Formal Languages — Exercise Sheet 11

Exercise 11.1
Let language $L = \{w \in \{a,b\}^\omega : w \text{ contains finitely many } a\}$

(a) Give a deterministic Rabin automaton for L.

(b) Give an NBA for L and try to "determinize" it by using the NFA to DFA powerset construction. What is the language accepted by the resulting DBA?

(c) What ω-language is accepted by the following Muller automaton with acceptance condition $\{\{q_0\},\{q_1\},\{q_2\}\}$? And with acceptance condition $\{\{q_0,q_1\},\{q_1,q_2\},\{q_2,q_0\}\}$?

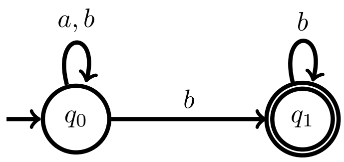


Exercise 11.2
Give a procedure that translates non-deterministic Rabin automata to non-deterministic Büchi automata.

Exercise 11.3
Let $L_\sigma = \{w \in \{a,b,c\}^\omega : w \text{ contains infinitely many } \sigma\text{'s}\}$. Give deterministic Büchi automata for languages L_a, L_b and L_c, and construct the intersection of these automata.

Exercise 11.4

(a) Consider the following Büchi automaton A over $\Sigma = \{a,b\}$:



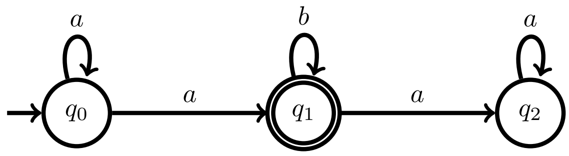
Draw $\text{dag}(abab^\omega)$ and $\text{dag}((ab)^\omega)$.

(b) Let r_w be the ranking of $\text{dag}(w)$ defined by

$$r_w(q,i) = \begin{cases} 1 & \text{if } q = q_0 \text{ and } \langle q_0, i\rangle \text{ appears in dag}(w), \\ 0 & \text{if } q = q_1 \text{ and } \langle q_1, i\rangle \text{ appears in dag}(w), \\ \bot & \text{otherwise.} \end{cases}$$

Are r_{abab^ω} and $r_{(ab)^\omega}$ (over A) odd rankings?

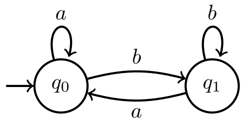
(c) Consider the following Büchi automaton B over $\Sigma = \{a, b\}$:



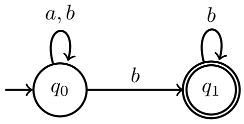
Draw dag(a^ω). Show that any odd ranking for this dag must contain a rank of 3 or more.

Solution 11.1

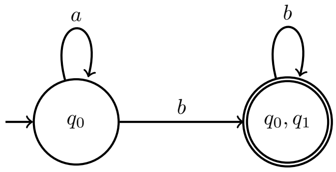
(a) The following DRA, with acceptance condition $\{\langle\{q_1\},\{q_0\}\rangle\}$, i.e., a run is accepting iff it visits q_1 infinitely often and q_0 finitely often, recognizes L:



(b) This NBA accepts L:



The powerset construction yields the DBA below (with the trap state omitted). It recognizes the language a^*b^ω, which is different from $(a+b)^*b^\omega$:



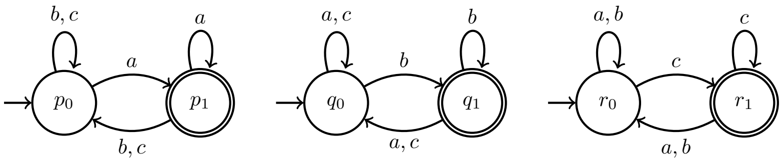
(c) With the first acceptance condition the language is $\Sigma^*(a^\omega + b^\omega + c^\omega)$. With the second, the automaton does not accept any word. Indeed, every run that visits both q_0 and q_1 infinitely often must also visit q_2 infinitely often, and the same holds for q_1 and q_2, and for q_2 and q_0.

Solution 11.2

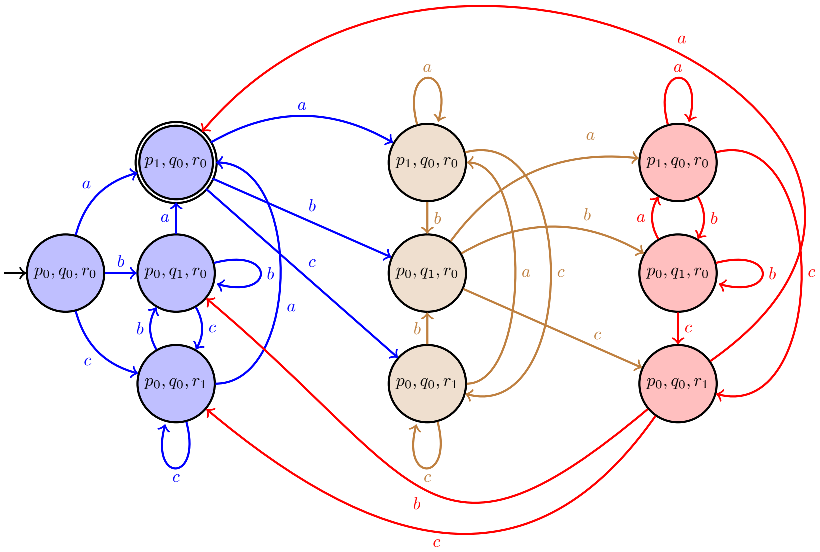
Given a Rabin automaton $A = \big(Q,\Sigma,Q_0,\delta,\big\{\langle F_0,G_0\rangle,\ldots,\langle F_{m-1},G_{m-1}\rangle\big\}\big)$, it follows easily that $L_\omega(A) = \bigcup_{i=0}^{m-1} L_\omega(A_i)$ where each $A_i = (Q,\Sigma,Q_0,\delta,\{\langle F_i,G_i\rangle\})$. So it suffices to translate each A_i into an NBA B_i and take the union of the B_i's. For this, we use the same idea that we used for converting an NCA into an NBA (as shown in the previous exercise sheet). To construct B_i, we take two copies of A_i, say A_i^0 and A_i^1, where A_i^0 is a full copy of A_i and A_i^1 is a partial copy containing only the states of $Q \setminus G_i$ and the transitions between these states. We let $[q,i]$ denote the i^{th} copy of the state q and for every transition $q \xrightarrow{a} q'$ in A_i with $q' \in Q \setminus G_i$, we add a transition $[q,0] \xrightarrow{a} [q',1]$ to B_i. We set the initial states to be $\{[q,0], q \in Q_0\}$ and we set the final states to be $\{[q,1] : q \in F_i\}$. Similar to the last exercise of the previous sheet, we can show that B_i accepts $L_\omega(A_i)$.

Solution 11.3

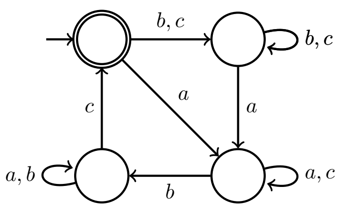
The following deterministic Büchi automata respectively accept L_a, L_b and L_c:



Taking their intersection leads to the following deterministic Büchi automaton:

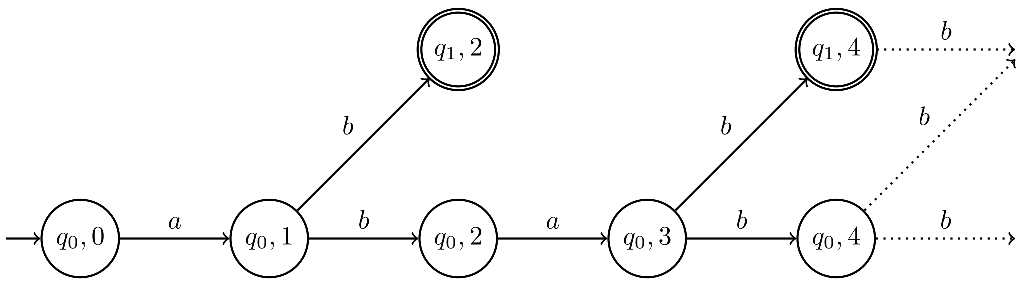


Note that $L_a \cap L_b \cap L_b$ is accepted by a smaller DBA:

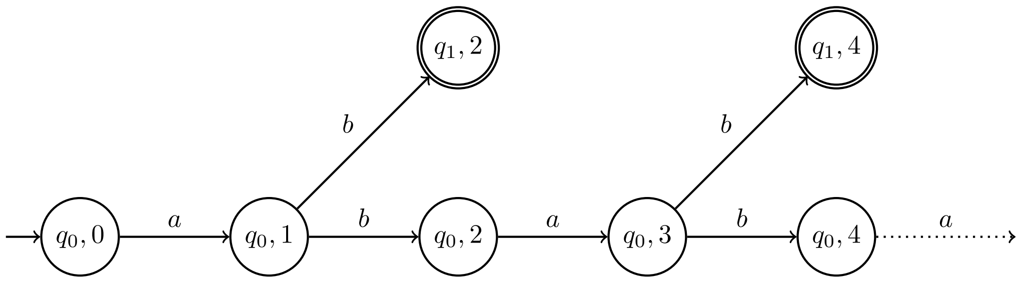


Solution 11.4

(a) dag($abab^\omega$):



dag($(ab)^\omega$):



(b) • r *is not* an odd rank for dag($abab^\omega$) since

$$\langle q_0, 0\rangle \xrightarrow{a} \langle q_0, 1\rangle \xrightarrow{b} \langle q_0, 2\rangle \xrightarrow{a} \langle q_0, 3\rangle \xrightarrow{b} \langle q_1, 4\rangle \xrightarrow{b} \langle q_1, 5\rangle \xrightarrow{b} \cdots$$

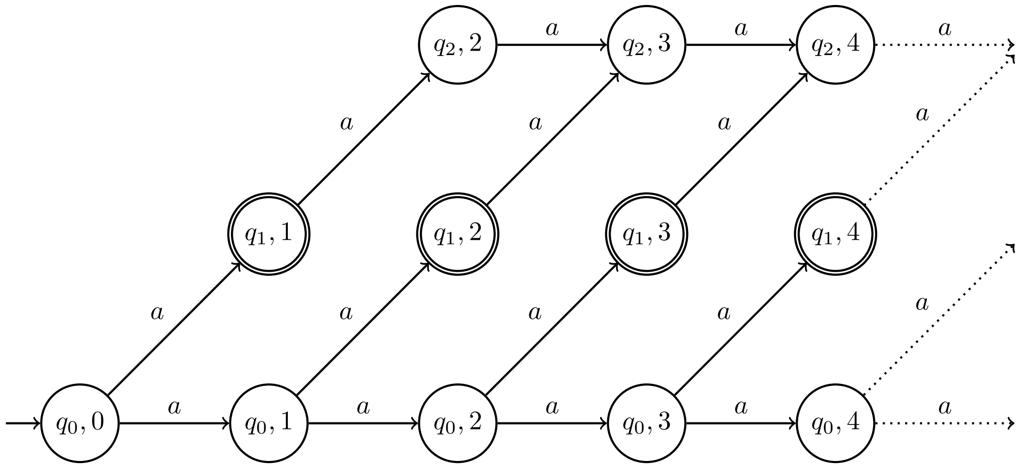
is an infinite path of dag($abab^\omega$) not visiting odd nodes infinitely often.

- r *is* an odd rank for $\text{dag}((ab)^\omega)$ since it has a single infinite path:

$$\langle q_0, 0\rangle \xrightarrow{a} \langle q_0, 1\rangle \xrightarrow{b} \langle q_0, 2\rangle \xrightarrow{a} \langle q_0, 3\rangle \xrightarrow{b} \langle q_0, 4\rangle \xrightarrow{a} \langle q_0, 5\rangle \xrightarrow{b} \cdots$$

  which only visits odd nodes.

(c) $\text{dag}(a^\omega)$:



Let r be an odd rank for $\text{dag}(a^\omega)$. It exists since a^ω is not accepted by B. Since r is odd, all infinite paths must visit odd nodes infinitely often (i.o.). In particular the bottom infinite path of q_0 nodes must stabilize to nodes with odd rank.

Let us assume the nodes $\langle q_0, j\rangle$ have rank 1 for all $j \geq i$ for some $i \geq 0$. Consider the infinite path $\rho = \langle q_0, i\rangle \xrightarrow{a} \langle q_1, i+1\rangle \xrightarrow{a} \langle q_2, i+2\rangle \xrightarrow{a} \langle q_2, i+3\rangle \ldots$. Node $\langle q_1, i+1\rangle$ must have an even rank (since q_1 is accepting) smaller or equal to 1, so it has rank 0. This entails that $\langle q_2, k\rangle$ has rank 0 for all $k \geq i+2$. This contradicts r being an odd ranking because the path ρ is infinite yet does not visit odd nodes infinitely often.

Thus the bottom infinite path of q_0 nodes must stabilize to nodes with odd rank strictly bigger than 1, i.e., bigger or equal to 3.