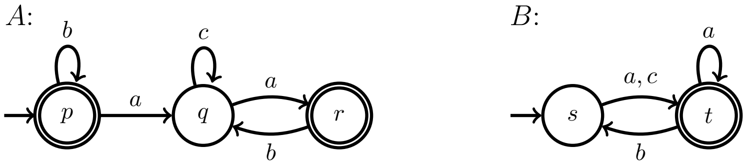
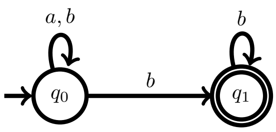
Automata and Formal Languages — Exercise Sheet 11

Exercise 11.1
Construct the intersection of the two following Büchi automata:



Exercise 11.2
Consider the following Büchi automaton B over $\Sigma = \{a, b\}$:



(a) Sketch $\text{dag}(abab^\omega)$ and $\text{dag}((ab)^\omega)$.

(b) Let r_w be the ranking of $\text{dag}(w)$ defined by

$$r_w(q, i) = \begin{cases} 1 & \text{if } q = q_0 \text{ and } \langle q_0, i \rangle \text{ appears in dag}(w), \\ 0 & \text{if } q = q_1 \text{ and } \langle q_1, i \rangle \text{ appears in dag}(w), \\ \perp & \text{otherwise.} \end{cases}$$

Are r_{abab^ω} and $r_{(ab)^\omega}$ odd rankings?

(c) Show that r_w is an odd ranking if and only if $w \notin L_\omega(B)$.

(d) Construct a Büchi automaton accepting $\overline{L_\omega(B)}$ using the construction seen in class. *Hint*: by (c), it is sufficient to use $\{0, 1\}$ as ranks.

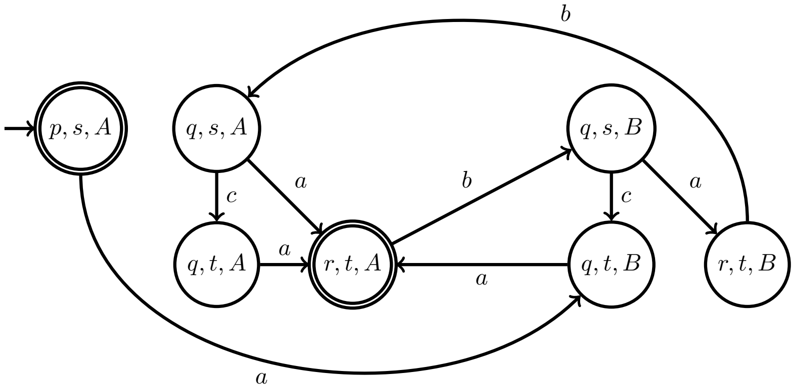
Exercise 11.3
Show that for every DBA A with n states there is an NBA B with $2n$ states such that $B = \overline{A}$. Explain why your construction does not work for NBAs.

Exercise 11.4
Give Büchi automata for the following ω-languages:

- $L_1 = \{w \in \{a, b\}^\omega : w \text{ contains infinitely many } a\text{'s}\}$,
- $L_2 = \{w \in \{a, b\}^\omega : w \text{ contains finitely many } b\text{'s}\}$,
- $L_3 = \{w \in \{a, b\}^\omega : \text{each occurrence of } a \text{ in } w \text{ is followed by a } b\}$,

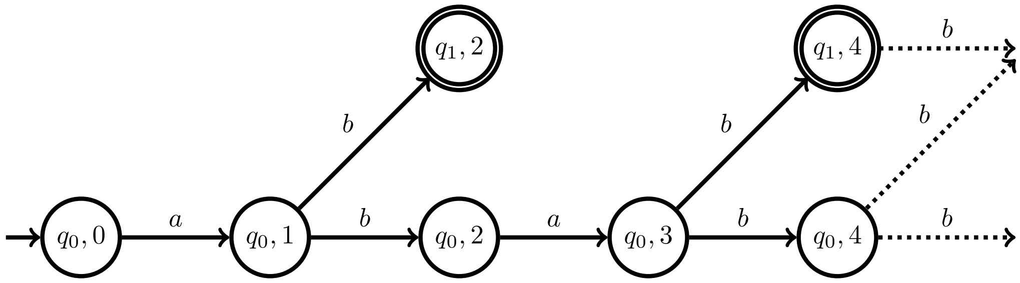
and intersect these automata. Decide if this automaton is the smallest Büchi automaton for that language.

Solution 11.1

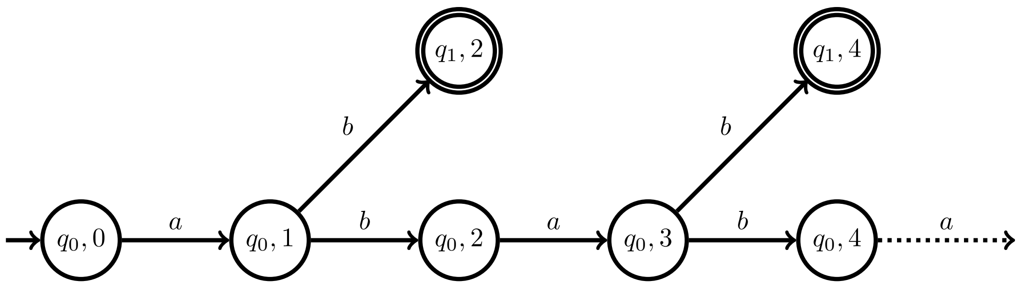


Solution 11.2

(a) $\mathrm{dag}(abab^{\omega})$:



dag($(ab)^\omega$):



(b)
- r *is not* an odd rank for dag($abab^\omega$) since

$$\langle q_0, 0\rangle \xrightarrow{a} \langle q_0, 1\rangle \xrightarrow{b} \langle q_0, 2\rangle \xrightarrow{a} \langle q_0, 3\rangle \xrightarrow{b} \langle q_1, 4\rangle \xrightarrow{b} \langle q_1, 5\rangle \xrightarrow{b} \cdots$$

  is an infinite path of dag($abab^\omega$) not visiting odd nodes infinitely often.

- r *is* an odd rank for dag($(ab)^\omega$) since it has a single infinite path:

$$\langle q_0, 0\rangle \xrightarrow{a} \langle q_0, 1\rangle \xrightarrow{b} \langle q_0, 2\rangle \xrightarrow{a} \langle q_0, 3\rangle \xrightarrow{b} \langle q_0, 4\rangle \xrightarrow{a} \langle q_0, 5\rangle \xrightarrow{b} \cdots$$

  which only visits odd nodes.

(c) $\Rightarrow$) Let $w \in L_\omega(B)$. We have $w = ub^\omega$ for some $u \in \{a, b\}^*$. This implies that

$$\langle q_0, 0\rangle \xrightarrow{u} \langle q_0, |u|\rangle \xrightarrow{b} \langle q_1, |u|+1\rangle \xrightarrow{b} \langle q_1, |u|+2\rangle \xrightarrow{b} \cdots$$

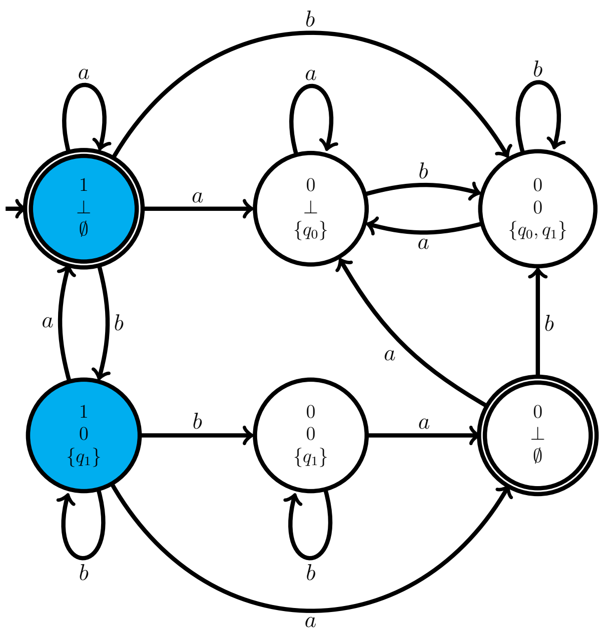
is an infinite path of dag(w). Since this path does not visit odd nodes infinitely often, r is not odd for dag(w).

$\Leftarrow$) Let $w \notin L_\omega(B)$. Suppose there exists an infinite path of dag(w) that does not visit odd nodes infinitely often. At some point, this path must only visit nodes of the form $\langle q_1, i\rangle$. Therefore, there exists $u \in \{a, b\}^*$ such that

$$\langle q_0, 0\rangle \xrightarrow{u} \langle q_1, |u|\rangle \xrightarrow{b} \langle q_1, |u|+1\rangle \xrightarrow{b} \langle q_1, |u|+2\rangle \xrightarrow{b} \cdots$$

This implies that $w = ub^\omega \in L_\omega(B)$ which is contradiction.

(d) Recall that we construct an NBA with an infinite number of states whose runs on an ω-word w are the rankings of *dag*(w). The automaton accepts a ranking R iff every infinite path of R visits nodes of odd rank i.o. By (c), for every $w \in \{a, b\}^\omega$, if dag(w) has an odd ranking, then it has one ranging over 0 and 1. Therefore, it suffices to execute *CompNBA* with rankings ranging over 0 and 1 (and our NBA is now finite). We obtain the following Büchi automaton, for which some intuition is given below:



Any ranking r of $dag(w)$ can be decomposed into a sequence $lr_1, lr_2, \ldots$ such that $lr_i(q) = r(< q, i >)$, the level i of rank r. Recall that in this automaton, the transitions $\begin{bmatrix} lr(q_0) \\ lr(q_1) \end{bmatrix} \xrightarrow{a} \begin{bmatrix} lr'(q_0) \\ lr'(q_1) \end{bmatrix}$ represent the possible next level for ranks r such that $lr(q) = r(< q, i >)$ and $lr'(q) = r(< q, i+1 >)$ for $q = q_0, q_1$.

The additional set of states in the automaton represents the set of states that "owe" a visit to a state of odd rank. Formally, the transitions are the triples $[lr, O] \xrightarrow{a} [lr', O']$ such that $lr \xrightarrow{a} lr'$ and $O' = \{q' \in \delta(O, a) | lr'(q') \textit{ is even}\}$ if $O \neq \emptyset$, and $O' = \{q' \in Q | lr'(q') \textit{ is even}\}$ if $O = \emptyset$.

Finally the accepting states of the automaton are those with no "owing" states, which represent the *breakpoints* i.e. a moment where we are sure that all runs on w have seen an odd rank since the last breakpoint.

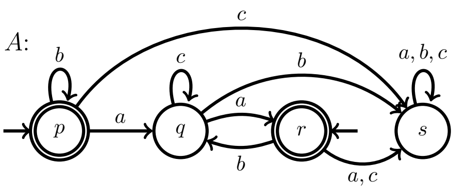
★ It is enough to only consider the blue states, as any other state cannot reach a level in which there is an odd rank; descendants of *dag* states with rank 0 can never be assigned an odd rank.

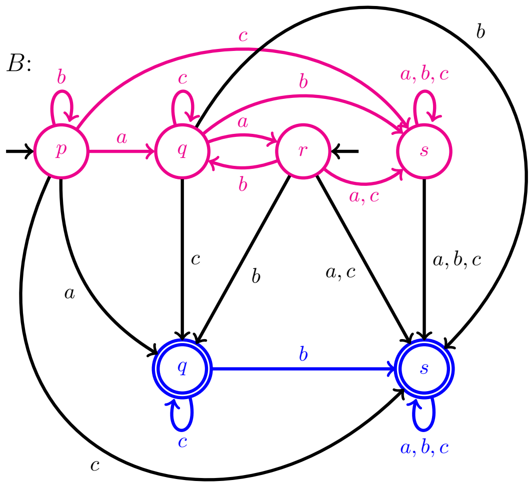
Solution 11.3

Observe that A rejects a word w iff its *single* run on w stops visiting accepting states at some point. Hence, we construct an NBA B that reads a prefix as in A and non deterministically decides to stop visiting accepting states by moving to a copy of A without its accepting states.

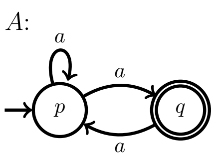
More precisely, we assume that each letter can be read from each state of A, i.e. that A is complete. If this is not the case, it suffices to add a rejecting sink state to A. The NBA B consists of two copies of A. The first copy is exactly as A. The second copy is as A but restricted to its non accepting states. We add transitions from the first copy to the second one as follows. For each transition (p, a, q) of A, we add a transition that reads letter a from state p of the first copy to state q of the second copy. All states of the first copy are made non accepting and all states of the second copy are made accepting. Note that B contains at most $2n$ states as desired.

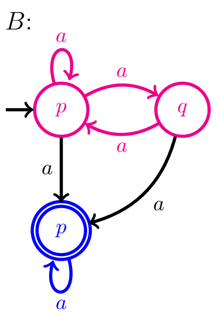
Here is an example of the construction:





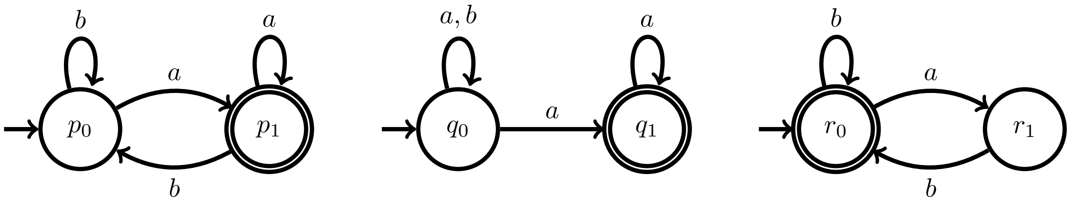
This construction does not work on NBAs. Indeed, we have $A = B = \{a^\omega\}$ below:



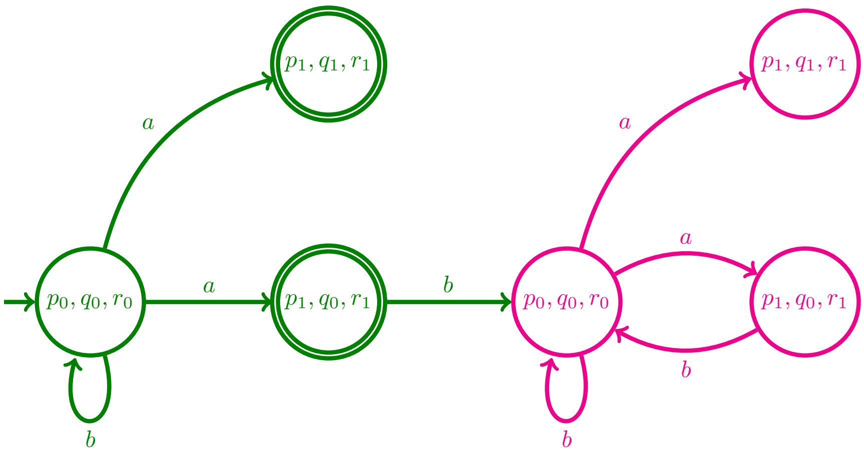


Solution 11.4

The following Büchi automata respectively accept L_1, L_2 and L_3:



Taking the intersection of these automata leads to the following Büchi automaton:



★ Note that the language of this automaton is the empty language.