Technische Universität München **Winter term 2021/22**
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Automata and Formal Languages — Exercise Sheet 10

Exercise 10.1
An ω-automaton has acceptance on transitions if the acceptance condition specifies which transitions must appear infinitely often in a run. All classes of ω-automata (Büchi, co-Büchi, etc.) can be defined with acceptance on transitions rather than states.

Give minimal deterministic automata for the language of words over $\{a, b\}$ containing infinitely many a and infinitely many b. of the following kinds: (a) Büchi, (b) generalized Büchi, (c) Büchi with acceptance on transitions, and (d) generalized Büchi with acceptance on transitions.

Exercise 10.2
The *limit* of a language $L \subseteq \Sigma^*$ is the ω-language $lim(L)$ defined as: $w \in lim(L)$ iff infinitely many prefixes of w are words of L, e.g. the limit of $(ab)^*$ is $(ab)^\omega$.

(a) Determine the limit of the following regular languages over $\{a, b\}$:

   (i) $(a + b)^* a^*$,

   (ii) $(bab)^* b$.

   (iii) $\{w : \text{ At least one } a \text{ appears in } w\}$

   (iv) $\{w : \text{ Number of appearances of } a \text{ in } w \text{ is odd and at least 3 }\}$

(b) Prove the following: An ω-language is recognizable by a deterministic Büchi automaton iff it is the limit of a regular language.

(c) Exhibit a non-regular language whose limit is ω-regular.

(d) Exhibit a non-regular language whose limit is not ω-regular.

Exercise 10.3
Let $L_1 = (ab)^\omega$ and let L_2 be the language of all words over $\{a, b\}$ containing infinitely many a and infinitely many b.

(a) Exhibit three different DBAs with three states recognizing L_1.

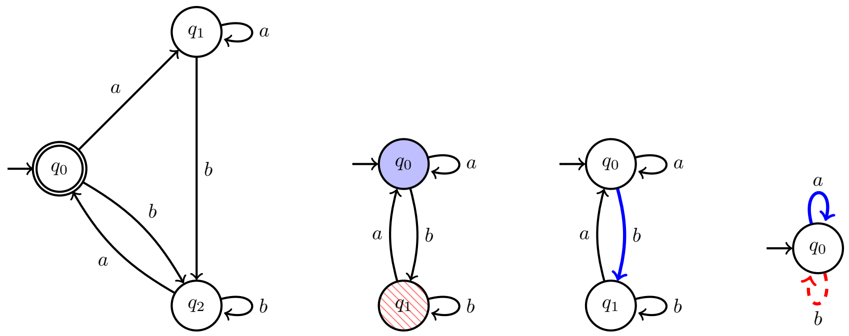
(b) Exhibit six different DBAs with three states recognizing L_2.

(c) Show that no DBA with at most two states recognizes L_1 or L_2.

Exercise 10.4
Show that for every NCA there is an equivalent NBA.

Solution 10.1

Automata (a), (b), (c) and (d) are respectively as follows, where colored patterns indicate the sets of accepting states or transitions:



Solution 10.2

(a) (i) $\{a, b\}^\omega$.

(ii) The empty ω-language.

(iii) The set of ω-words containing infinitely many a.

(iv) The set of ω-words containing infinitely many a, plus the set of ω-words such that the number of a's appearing in them is finite, odd and bigger than 3.

(b) Let B be a DFA recognizing L'. Consider B as a DBA, and let L be the ω-language recognized by B. We show that $L = \mathit{lim}(L')$. If $w \in \mathit{lim}(L')$, then B (as a DFA) accepts infinitely many prefixes of w. Since B is deterministic, the runs of B on these prefixes are prefixes of the unique infinite run of B (as a DBA) on w. So the infinite run visits accepting states infinitely often, and so $w \in L$. If $w \in L$, then the unique run of B on w (as a DBA) visits accepting states infinitely often, and so infinitely many prefixes of w are accepted by B (as a DFA). Thus, $w \in \mathit{lim}(L')$.

If L is the limit of a regular language L', then by the above argument, it is clear that L is an ω-language recognizable by a DBA.

Suppose L is an ω-language recognizable by a DBA (say B). Consider B as a DFA and let L' be the language recognized by it. By the above argument, it is clear that $L = \mathit{lim}(L')$ and so L is the limit of a regular language.

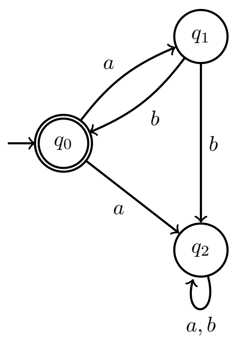
□

(c) Let $L = \{a^n b^n : n \geq 0\}$, which is not a regular language. Then $\mathit{lim}(L) = \emptyset$, which is ω-regular.

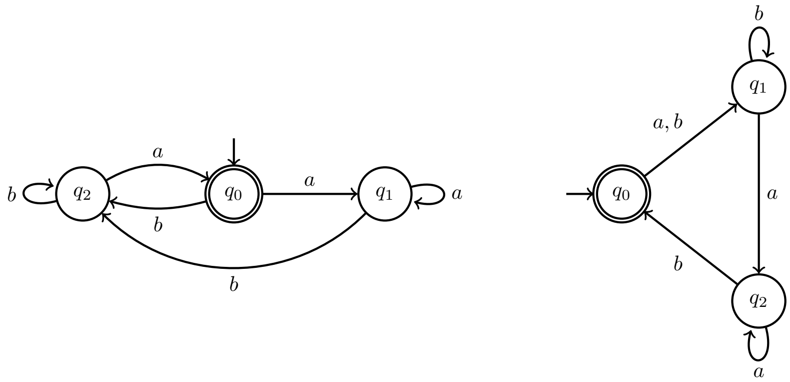
(d) Let $L = \{a^n b^n c^m : n, m \geq 0\}$. We have $\mathit{lim}(L) = \{a^n b^n c^\omega : n \geq 0\}$. Suppose this language is ω-regular and hence recognized by a Büchi automaton B. By the pigeonhole principle, there are distinct $n_1, n_2 \in \mathbb{N}$ and accepting runs ρ_1, ρ_2 of B on $a^{n_1} b^{n_1} c^\omega$ and $a^{n_2} b^{n_2} c^\omega$ such that the state reached in ρ_1 after reading a^{n_1} and the state eached in ρ_2 after reading a^{n_2} coincide. This means that B accepts $a^{n_1} b^{n_2} c^\omega$, which contradicts the assumption that B recognizes L.

Solution 10.3

(a) We obtain three DBAs for L_1 from the one below by making either q_0, q_1 or both accepting:



(b) Here are two different DBAs for L_2. We obtain two further DBAs from each of these automata by making either q_1 or q_2 the initial state.



(c) Assume there is a DBA B with at most two states recognizing L_1. Since L_1 is nonempty, B has at least one (reachable) accepting state q. Consider the transitions leaving q labeled by a and b. If any of them leads to q again, then B accepts an ω-word of the form wa^ω or wb^ω for some finite word w. Since no word of this form belongs to L_1, we reach a contradiction. Thus, B must have two states q and q', and transitions

$$t_a = q \xrightarrow{a} q' \text{ and } t_b = q \xrightarrow{b} q'.$$

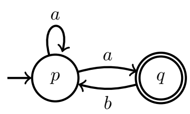
Consider any accepting run ρ of B. If the word accepted by the run does not belong to L_1, we are done. So assume it belongs to L_1. Since ρ is accepting, it contains some occurrence of t_a or t_b. Consider the run ρ' obtained by exchanging the first occurrence of one of them by the other (that is, if t_a occurs first, then replace it by t_b, and vice versa). Then ρ' is an accepting run, and the word it accepts is the result of turning an a into a b, or vice versa. In both cases, the resulting word does not belong to L_1; so we each again a contradiction, and we are done.

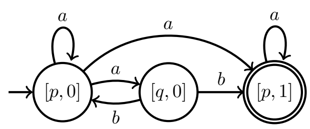
The proof for L_2 is similar. □

Solution 10.4

Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NCA. We construct an NBA B which is equivalent to A. Observe that the co-Büchi accepting condition $\inf(\rho) \cap F = \emptyset$ is equivalent to $\inf(\rho) \subseteq Q \setminus F$. This condition holds iff ρ has an infinite suffix that only visits states of $Q \setminus F$. We design B in two stages. In the first one, we take two copies of A, that we call A_0 and A_1, and put them side by side; A_0 is a full copy, containing all states and transitions of A, and A_1 is a partial copy, containing only the states of $Q \setminus F$ and the transitions between these states. We write $[q, 0]$ to denote the copy a state $q \in Q$ in A_0, and $[q, 1]$ for the copy of state $q \in Q \setminus F$ in A_1. In the second stage, we add some transitions that "jump" from A_0 to A_1: for every transition $[q, 0] \xrightarrow{a} [q', 0]$ of A_0 such that $q' \in Q \setminus F$, we add a transition $[q, 0] \xrightarrow{a} [q', 1]$ that "jumps" to $[q', 1]$, the "twin state" of $[q', 0]$ in A_1. Note that $[q, 0] \xrightarrow{a} [q', 1]$ does not replace $[q, 0] \xrightarrow{a} [q', 0]$, it is an *additional* transition. As initial states of B, we choose the copy of Q_0 in A_0, i.e., $\{[q, 0] : q \in Q_0\}$, and as accepting states all the states of A_1, i.e., $\{[q, 1] : q \in Q \setminus F\}$.

For example, the NCA below on the left is transformed into the NBA on the right:





It remains to show that $L_\omega(A) = L_\omega(B)$.

$\subseteq$) Let $w \in L_\omega(A)$. There is a run ρ of A on word w such that $\inf \rho \cap F = \emptyset$. It follows that $\rho = \rho_0\, \rho_1$, where ρ_0 is a finite prefix of ρ, and ρ_1 is an infinite suffix that only contains states of $Q \setminus F$. Let ρ' be the run of B on w that simulates ρ_0 on A_0, and then "jumps" to A_1 and simulates ρ_1 in A_1. Notice that ρ' exists because

ρ_1 only visits states of $Q \setminus F$. Since all states of A_1 are accepting, ρ' is an accepting run of the NBA B, and so $w \in L_\omega(B)$.

$\supseteq$) Let $w \in L_\omega(B)$. There is an accepting run ρ of B on word w. Thus, ρ visits states of A_1 infinitely often. Since a run of B that enters A_1 can never return to A_0 (there are no "back-jumps" from A_1 to A_0,) ρ has an infinite suffix ρ_1 that only visits states of A_1, i.e., states $[q, 1]$ such that $q \in Q \setminus F$. Let ρ' be the result of replacing $[q, 0]$ and $[q, 1]$ by q everywhere in ρ. Clearly, ρ' is a run of A on w that only visits F finitely often. Thus, ρ' is an accepting run of A, and $w \in \mathcal{L}A$. $\square$