Automata and Formal Languages — Exercise Sheet 9

Exercise 9.1

Consider the logic PureMSO(Σ) with syntax

$$\varphi := X \subseteq Q_a \mid X < Y \mid X \subseteq Y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists X.\ \varphi$$

Notice that formulas of PureMSO(Σ) do not contain first-order variables. The satisfaction relation of PureMSO(Σ) is given by:

$$\begin{array}{llll} (w, \mathcal{J}) & \models & X \subseteq Q_a & \text{iff} \quad w[p] = a \text{ for every } p \in \mathcal{J}(X) \\ (w, \mathcal{J}) & \models & X < Y & \text{iff} \quad p < p' \text{ for every } p \in \mathcal{J}(X),\ p' \in \mathcal{J}(Y) \\ (w, \mathcal{J}) & \models & X \subseteq Y & \text{iff} \quad p \in \mathcal{J}(Y) \text{ for every } p \in \mathcal{J}(X) \end{array}$$

with the rest as for MSO(Σ).

Prove that MSO(Σ) and PureMSO(Σ) have the same expressive power for sentences. That is, show that for every sentence ϕ of MSO(Σ) there is an equivalent sentence ψ of PureMSO(Σ), and vice versa.

Exercise 9.2

Let $r \geq 0$ and $n \geq 1$. Give a Presburger formula φ such that $\mathcal{J} \models \varphi$ iff $\mathcal{J}(x) > \mathcal{J}(y)$ and $\mathcal{J}(x) - \mathcal{J}(y) \equiv r \pmod{n}$. Give an automaton that accepts the solutions of φ for $r = 1$ and $n = 2$.

Exercise 9.3

Let $\inf(w)$ denote the set of letters occurring infinitely often in the infinite word w. Give Büchi automata for the following ω-languages:

(a) $L_1 = \{w \in \Sigma^\omega : \text{in } w, \text{ every } a \text{ is immediately followed by a } b\}$ over alphabet $\Sigma = \{a, b, c\}$,

(b) $L_2 = \{w \in \Sigma^\omega : w \text{ has no occurrence of } bab\}$ over alphabet $\Sigma = \{a, b\}$,

(c) $L_3 = \{w \in \Sigma^\omega : \inf(w) \subseteq \{a, b\}\}$ over alphabet $\Sigma = \{a, b, c\}$,

(d) $L_4 = \{w \in \Sigma^\omega : \{a, b\} \subseteq \inf(w)\}$ over alphabet $\Sigma = \{a, b, c\}$,

(e) Prove that there is no deterministic Büchi automaton accepting L_3.

Exercise 9.4

Prove or disprove:

(a) For every Büchi automaton A, there exists a Büchi automaton B with a single initial state such that $L_\omega(A) = L_\omega(B)$.

(b) For every Büchi automaton A, there exists a Büchi automaton B with a single accepting state such that $L_\omega(A) = L_\omega(B)$.

(c) There exists a Büchi automaton recognizing the finite ω-language $\{w\}$ such that $w \in \{0, 1, \ldots, 9\}^\omega$ and w_i is the i^{th} decimal digit of π.

Solution 9.1

Given a sentence ψ of PureMSO(Σ), let ϕ be the sentence of MSO(Σ) obtained by replacing every subformula of ψ of the form

$$\begin{aligned} X \subseteq Y \quad & \text{by} \quad \forall x\ (x \in X \rightarrow x \in Y) \\ X \subseteq Q_a \quad & \text{by} \quad \forall x\ (x \in X \rightarrow Q_a(x)) \\ X < Y \quad & \text{by} \quad \forall x\ \forall y\ (x \in X \land y \in Y) \rightarrow x < y \end{aligned}$$

Clearly, ϕ and ψ are equivalent. For the other direction, let

$$\text{empty}(X) := \forall Y\ X \subseteq Y$$

and

$$\text{sing}(X) := \neg\text{empty}(X) \land \forall Y\ (Y \subseteq X) \rightarrow (\text{empty}(Y) \lor Y = X).$$

Intuitively, empty(X) is true iff X is the empty set and sing(X) is true iff X is a set of size one.

Let ϕ be a sentence of MSO(Σ). Assume without loss of generality that for every first-order variable x the second-order variable X does not appear in ϕ (if necessary, rename second-order variables appropriately). Let ψ be the sentence of PureMSO(Σ) obtained by replacing every subformula of ϕ of the form

$\exists x\ \psi'$	by	$\exists X\ (\text{sing}(X) \land \psi'[X/x])$
		where $\psi'[X/x]$ is the result of substituting X for x in ψ'
$Q_a(x)$	by	$X \subseteq Q_a$
$x < y$	by	$X < Y$
$x \in Y$	by	$X \subseteq Y$

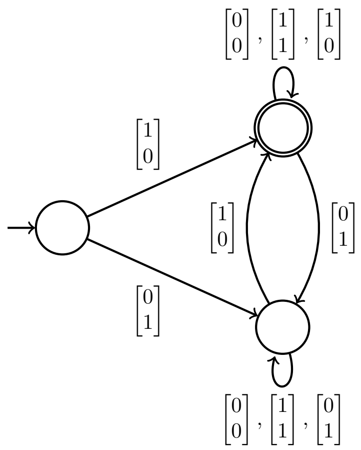
Clearly, ϕ and ψ are equivalent.

Solution 9.2

Let $0 \leq r' < n$ such that $r' \equiv r(\text{mod} n)$. Since n and r are fixed constants, r' is also a fixed constant. Further, since n is a constant, we can multiply a variable by n via iterated addition. The required formula is then given by:

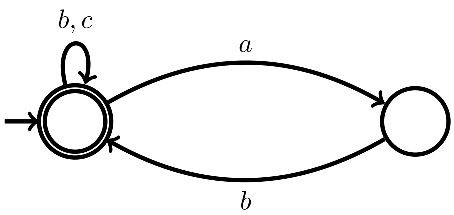
$$\varphi(x, y) := (x > y) \land \exists a\ (x = y + n \cdot a + r').$$

Let $k \in \mathbb{N}$ and $x, y \in \Sigma^k$ be LSBF encodings of some naturals. First note that $\text{val}(x) - \text{val}(y) \equiv 1 \pmod 2$ iff val(x) and val(y) are such that one is odd and the other one is even. Thus, the first bit of x and y should be different. Moreover, $\text{val}(x) > \text{val}(y)$ iff there exists $\ell \in \{1, \ldots, k\}$ such that $x_\ell = 1$, $y_\ell = 0$, and $x_i \geq y_i$ for every $\ell < i \leq k$. These observations yield the following automaton:

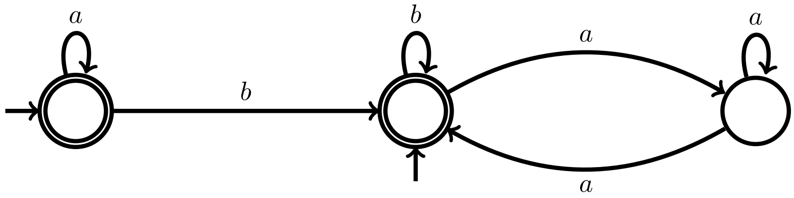


Solution 9.3

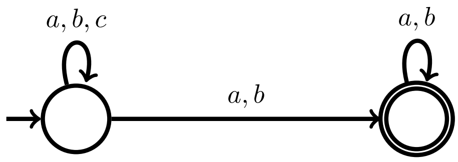
These are just some possible solutions.



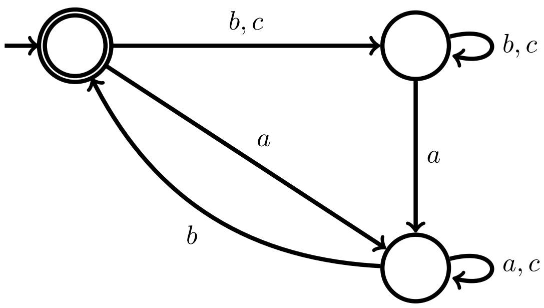
(a)



(b)

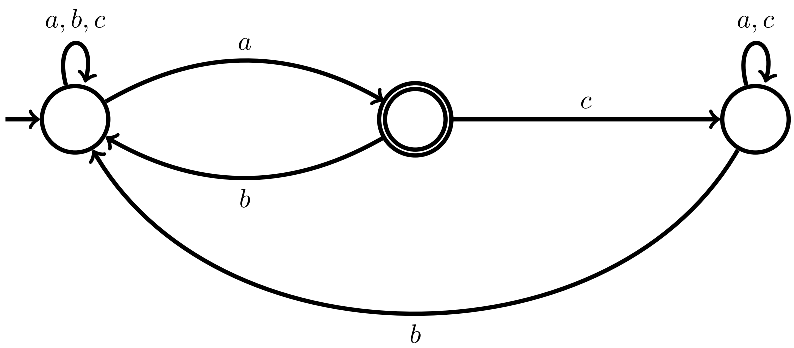


(c)



(d)

or



(e) For the sake of contradiction, suppose there exists a deterministic Büchi automaton $B = (Q, \Sigma, \delta, q_0, F)$ such that $L_\omega(B) = L_3$. Since $cb^\omega \in L_3$, B must visit F infinitely often when reading cb^ω. In particular, this implies the existence of $m_1 > 0$ and $q_1 \in F$ such that $q_0 \xrightarrow{cb^{m_1}} q_1$. Similarly, since $cb^{m_1}cb^\omega \in L_3$, there exist $m_2 > 0$ and $q_2 \in F$ such that $q_0 \xrightarrow{cb^{m_1}cb^{m_2}} q_2$. Since B is deterministic, we have $q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2$. By repeating this argument $|Q|$ times, we can construct $m_1, m_2, \ldots, m_{|Q|} > 0$ and $q_1, q_2, \ldots, q_{|Q|} \in F$ such that

$$q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2 \cdots \xrightarrow{cb^{m_{|Q|}}} q_{|Q|}.$$

By the pigeonhole principle, there exist $0 \leq i < j \leq |Q|$ such that $q_i = q_j$. Let

$$u = cb^{m_1}cb^{m_2} \cdots cb^{m_i},$$
$$v = cb^{m_{i+1}}cb^{m_{i+2}} \cdots cb^{m_j}.$$

We have $q_0 \xrightarrow{u} q_i \xrightarrow{v} q_i \xrightarrow{v} q_i \xrightarrow{v} \cdots$ which implies that $uv^\omega \in L_\omega(B)$. Also notice that c appears infinitely often in uv^ω, that is, $c \in \inf(uv^\omega)$. Therefore we have $uv^\omega \notin L_3 = L_\omega(B)$, which yields a contradiction. □

Solution 9.4

(a) True. The construction for NFAs still work for Büchi automata.

Let $B = (Q, \Sigma, \delta, Q_0, F)$ be a Büchi automaton. We add a state to Q which acts as the single initial state. More formally, we define $B' = (Q \cup \{q_{\text{init}}\}, \Sigma, \delta', \{q_{\text{init}}\}, F)$ where

$$\delta'(q, a) = \begin{cases} \bigcup_{q_0 \in Q_0} \delta(q_0, a) & \text{if } q = q_{\text{init}}, \\ \delta(q, a) & \text{otherwise.} \end{cases}$$

We have $L_\omega(B) = L_\omega(B')$, since there exists $q_0 \in Q_0$ such that

$$q_0 \xrightarrow{a_1}_B q_1 \xrightarrow{a_2}_B q_2 \xrightarrow{a_3}_B \cdots$$

if and only if

$$q_{\text{init}} \xrightarrow{a_1}_{B'} q_1 \xrightarrow{a_2}_{B'} q_2 \xrightarrow{a_3}_{B'} \cdots$$

(b) False. Let $L = \{a^\omega, b^\omega\}$. Suppose there exists a Büchi automaton $B = (Q, \{a, b\}, \delta, Q_0, F)$ such that $L_\omega(B) = L$ and $F = \{q\}$. Since $a^\omega \in L$, there exist $q_0 \in Q_0$, $m \geq 0$ and $n > 0$ such that

$$q_0 \xrightarrow{a^m} q \xrightarrow{a^n} q.$$

Similarly, since $b^\omega \in L$, there exist $q_0' \in Q_0$, $m' \geq 0$ and $n' > 0$ such that

$$q_0' \xrightarrow{b^{m'}} q \xrightarrow{b^{n'}} q.$$

This implies that

$$q_0 \xrightarrow{a^m} q \xrightarrow{b^{n'}} q \xrightarrow{b^{n'}} \cdots$$

Therefore, $a^m(b^{n'})^\omega \in L$, which is a contradiction. □

(c) False. Suppose there exists a Büchi automaton $B = (Q, \{0, 1, \ldots, 9\}, \delta, Q_0, F)$ such that $L_\omega(B) = \{w\}$. There exist $u \in \{0, 1, \ldots, 9\}^*$, $v \in \{0, 1, \ldots, 9\}^+$, $q_0 \in Q_0$ and $q \in F$ such that

$$q_0 \xrightarrow{u} q \xrightarrow{v} q.$$

Therefore, $uv^\omega \in L_\omega(B)$ which implies that $w = uv^\omega$. Since w is the decimal representation of π, we conclude that π is rational, which is a contradiction. □