Automata and Formal Languages — Exercise Sheet 9

Exercise 9.1

(a) Give a formula Block_between of $\mathrm{MSO}(\Sigma)$ such that $\mathrm{Block\_between}(X, i, j)$ holds whenever $X = \{i, i+1, \ldots, j\}$.

(b) Let $0 \leq m < n$. Give a formula $\mathrm{Mod}^{m,n}$ of $\mathrm{MSO}(\Sigma)$ such that $\mathrm{Mod}^{m,n}(i, j)$ holds whenever $|w_i w_{i+1} \cdots w_j| \equiv m \pmod{n}$, i.e. whenever $j - i + 1 \equiv m \pmod{n}$.

(c) Let $0 \leq m < n$. Give a sentence of $\mathrm{MSO}(\Sigma)$ that defines $a^m(a^n)^*$.

(d) Give a sentence of $\mathrm{MSO}(\{a, b\})$ that defines the language of words such that every two b's with no other b in between are separated by a block of a's of odd length.

Exercise 9.2

Let $\mathrm{inf}(w)$ denote the set of letters occurring infinitely often in the infinite word w. Give Büchi automata and ω-regular expressions for the following ω-languages:

(a) $L_1 = \{w \in \Sigma^\omega : \text{in } w, \text{ every } a \text{ is immediately followed by a } b\}$ over alphabet $\Sigma = \{a, b, c\}$,

(b) $L_2 = \{w \in \Sigma^\omega : w \text{ has no occurrence of } bab\}$ over alphabet $\Sigma = \{a, b\}$,

(c) $L_3 = \{w \in \Sigma^\omega : \mathrm{inf}(w) \subseteq \{a, b\}\}$ over alphabet $\Sigma = \{a, b, c\}$,

(d) $L_4 = \{w \in \Sigma^\omega : \{a, b\} \subseteq \mathrm{inf}(w)\}$ over alphabet $\Sigma = \{a, b, c\}$,

(e) Does there exist a deterministic Büchi automaton accepting L_3? If there is then give it, otherwise give a proof of why it is not true.

Exercise 9.3

Prove or disprove:

(a) For every Büchi automaton A, there exists a Büchi automaton B with a single initial state and such that $L_\omega(A) = L_\omega(B)$;

(b) For every Büchi automaton A, there exists a Büchi automaton B with a single accepting state and such that $L_\omega(A) = L_\omega(B)$;

(c) There exists a Büchi automaton recognizing the finite ω-language $\{w\}$ such that $w \in \{0, 1, \ldots, 9\}^\omega$ and w_i is the i^{th} decimal of $\sqrt{2}$.

Exercise 9.4

Recall that every finite set of finite words is a regular language. Prove that this does not hold for infinite words. More precisely:

(a) Prove that every nonempty ω-regular language contains an *ultimately periodic* ω-word, i.e., an ω-word of the form uv^ω for some finite words $u \in \Sigma^*$ and $v \in \Sigma^+$.

(b) Give an ω-word w such that $\{w\}$ is not an ω-regular language.

Hint: use (a).

Solution 9.1

(a) $\text{Block\_between}(X, i, j) := \forall x\ (x \in X) \leftrightarrow (i \le x \land x \le j)$.

(b) The idea is to construct the set of positions $\{i + m + k \cdot n \mid k \ge 0\}$ and check if $j + 1$ is in the set.
$\text{Mod}^{m,n}(i, j) := \exists x\ \exists y\ (x = m + i \land y = j + 1 \land \text{Mult}^n(x, y))$ where

$$\text{Mult}^n(a, b) := \exists X\ (b \in X) \land \forall x\ (x \in X) \leftrightarrow [(x = a) \lor (\exists y \in X\ x = y + n)].$$

(c) $\underbrace{[(m = 0) \land (\neg \exists x\ \text{first}(x))]}_{\text{empty word if m=0}} \lor [\forall x\ Q_a(x) \land \exists x \exists y\ \text{first}(x) \land \text{last}(y) \land \text{Mod}^{m,n}(x, y)]$.

(d) $\forall x \forall y\ [\varphi(x, y) \to \psi(x, y)]$ where

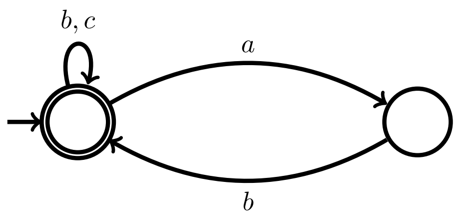
$$\varphi(x, y) := (x < y) \land Q_b(x) \land Q_b(y) \land [\forall z\ (x < z \land z < y) \to \neg Q_b(z)],$$
$$\psi(x, y) := [\forall z\ (x < z \land z < y) \to Q_a(z)] \land \text{Mod}^{1,2}(x, y).$$

In fact, since the question states that the alphabet is $\{a, b\}$, we can remove $\forall z\ (x < z \land z < y) \to Q_a(z)$ from $\psi(x, y)$.

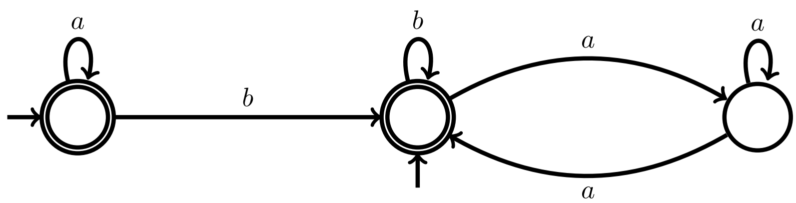
Solution 9.2

These are just some possible solutions.

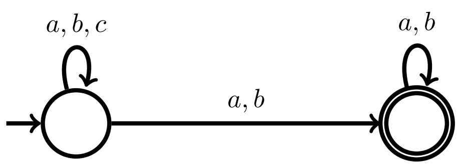
(a) $((b + c)^*(ab)^*)^\omega$, and



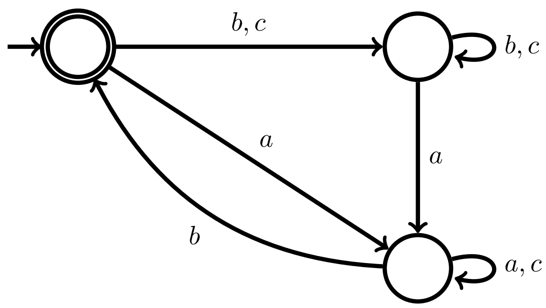
(b) $a^*(b^*(\epsilon + aaa^*))^\omega$, and



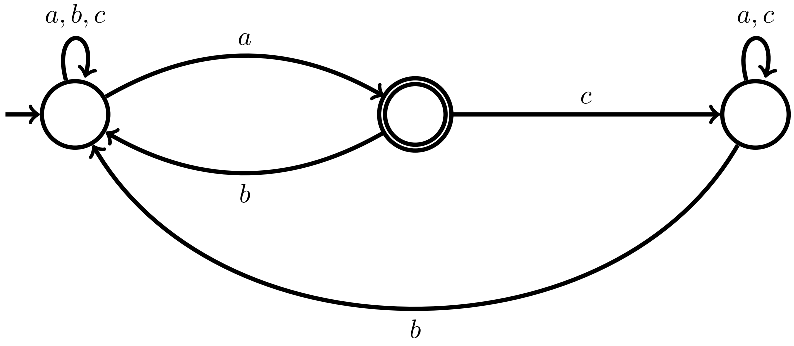
(c) $(a + b + c)^*(a + b)^\omega$, and



(d) $((b + c)^* a (a + c)^* b)^\omega$, and



or



(e) It is asked whether there exists a deterministic Büchi automaton accepting L_3. We show that it is *not* the case. For the sake of contradiction, suppose there exists a deterministic Büchi automaton $B = (Q, \Sigma, \delta, q_0, F)$ such that $L_\omega(B) = L_3$. Since $cb^\omega \in L_3$, B must visit F infinitely often when reading cb^ω. In particular, this implies the existence of $m_1 > 0$ and $q_1 \in F$ such that $q_0 \xrightarrow{cb^{m_1}} q_1$. Similarly, since $cb^{m_1}cb^\omega \in L_3$, there exist $m_2 > 0$ and $q_2 \in F$ such that $q_0 \xrightarrow{cb^{m_1}cb^{m_2}} q_2$. Since B is deterministic, we have $q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2$. By repeating this argument $|Q|$ times, we can construct $m_1, m_2, \ldots, m_{|Q|} > 0$ and $q_1, q_2, \ldots, q_{|Q|} \in F$ such that

$$q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2 \cdots \xrightarrow{cb^{m_{|Q|}}} q_{|Q|}.$$

By the pigeonhole principle, there exist $0 \leq i < j \leq |Q|$ such that $q_i = q_j$. Let

$$u = cb^{m_1}cb^{m_2} \cdots cb^{m_i},$$
$$v = cb^{m_{i+1}}cb^{m_{i+2}} \cdots cb^{m_j}.$$

We have $q_0 \xrightarrow{u} q_i \xrightarrow{v} q_i \xrightarrow{v} q_i \xrightarrow{v} \cdots$ which implies that $uv^\omega \in L_\omega(B)$. Also notice that c appears infinitely often in uv^ω, that is, $c \in \inf(uv^\omega)$. Therefore we have $uv^\omega \notin L_3 = L_\omega(B)$, which yields a contradiction. □

Solution 9.3

(a) True. The construction for NFAs still work for Büchi automata.

Let $B = (Q, \Sigma, \delta, Q_0, F)$ be a Büchi automaton. We add a state to Q which acts as the single initial state. More formally, we define $B' = (Q \cup \{q_{\text{init}}\}, \Sigma, \delta', \{q_{\text{init}}\}, F)$ where

$$\delta'(q, a) = \begin{cases} \bigcup_{q_0 \in Q_0} \delta(q_0, a) & \text{if } q = q_{\text{init}}, \\ \delta(q, a) & \text{otherwise.} \end{cases}$$

We have $L_\omega(B) = L_\omega(B')$, since there exists $q_0 \in Q_0$ such that

$$q_0 \xrightarrow{a_1}_B q_1 \xrightarrow{a_2}_B q_2 \xrightarrow{a_3}_B \cdots$$

if and only if

$$q_{\text{init}} \xrightarrow{a_1}_{B'} q_1 \xrightarrow{a_2}_{B'} q_2 \xrightarrow{a_3}_{B'} \cdots$$

(b) False. Let $L = \{a^\omega, b^\omega\}$. Suppose there exists a Büchi automaton $B = (Q, \{a, b\}, \delta, Q_0, F)$ such that $L_\omega(B) = L$ and $F = \{q\}$. Since $a^\omega \in L$, there exist $q_0 \in Q_0$, $m \geq 0$ and $n > 0$ such that

$$q_0 \xrightarrow{a^m} q \xrightarrow{a^n} q.$$

Similarly, since $b^\omega \in L$, there exist $q_0' \in Q_0$, $m' \geq 0$ and $n' > 0$ such that

$$q_0' \xrightarrow{b^{m'}} q \xrightarrow{b^{n'}} q.$$

This implies that

$$q_0 \xrightarrow{a^m} q \xrightarrow{b^{n'}} q \xrightarrow{b^{n'}} \cdots$$

Therefore, $a^m(b^{n'})^\omega \in L$, which is a contradiction. □

(c) False. Suppose there exists a Büchi automaton $B = (Q, \{0, 1, \ldots, 9\}, \delta, Q_0, F)$ such that $L_\omega(B) = \{w\}$. There exist $u \in \{0, 1, \ldots, 9\}^*$, $v \in \{0, 1, \ldots, 9\}^+$, $q_0 \in Q_0$ and $q \in F$ such that

$$q_0 \xrightarrow{u} q \xrightarrow{v} q.$$

Therefore, $uv^\omega \in L_\omega(B)$ which implies that $w = uv^\omega$. Since w represents the decimals of $\sqrt{2}$, we conclude that $\sqrt{2}$ is rational, which is a contradiction. □

Solution 9.4

(a) Let L be a nonempty ω-regular language and let $B = (Q, \{0, 1\}, \delta, Q_0, F)$ be an NBA that recognizes L. Since Q is finite, there exist $u \in \Sigma^*$, $v \in \Sigma^+$, $q_0 \in Q_0$ and $q \in F$ such that

$$q_0 \xrightarrow{u} q \xrightarrow{v} q.$$

Consequently, we have $uv^\omega \in L$ by iterating v from state q. □

(b) Let $w \in \{0, 1\}^\omega$ be the word given by

$$w_i = \begin{cases} 1 & \text{if } i \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

We prove that w is not ultimately periodic, which, by (a), implies that $\{w\}$ is not ω-regular. For the sake of contradiction, suppose $w = uv^\omega$ for some $u \in \{0, 1\}^*$ and $v \in \{0, 1\}^+$. If $v \in 0^*$, then we obtain a contradiction. Thus, there exists $1 \leq i \leq |v|$ such that $v_i = 1$. Let $m = |u| + i$ and $n = |v|$. By definition of w, $m + j \cdot n$ is a square for every $j \geq 0$. In particular, there exist $0 < a < b$ such that

$$m + n \cdot n = a^2 \text{ and } m + n \cdot n + n = b^2.$$

Note that $a \geq n$. Moreover,

$$b^2 = a^2 + n \leq a^2 + a < a^2 + 2a + 1 = (a + 1)^2.$$

Therefore $a^2 < b^2 < (a + 1)^2$ which is a contradiction. □