Automata and Formal Languages — Exercise Sheet 9

Exercise 9.1
Find ω-regular expressions (the shorter the better) for the following languages:

(1) $\{w \in \{a,b\}^\omega \mid k$ is even for each subword ba^kb of $w\}$

(2) $\{w \in \{a,b\}^\omega \mid w$ has no occurrence of $bab\}$

Exercise 9.2
Give *deterministic* Büchi automata recognizing the following ω-languages over $\Sigma = \{a,b,c\}$:

(a) $L_1 = \{w \in \Sigma^\omega : w$ contains at least one $c\}$,

(b) $L_2 = \{w \in \Sigma^\omega :$ in w, every a is immediately followed by a $b\}$,

(c) $L_3 = \{w \in \Sigma^\omega :$ in w, between two successive a's there are at least two b's$\}$.

Exercise 9.3
Let $\inf(w)$ denote the set of letters occurring infinitely often in the infinite word w. Give Büchi automata and ω-regular expressions for the following ω-languages over $\Sigma = \{a,b,c\}$:

(a) $L_1 = \{w \in \Sigma^\omega : \inf(w) \subseteq \{a,b\}\}$,

(b) $L_2 = \{w \in \Sigma^\omega : \inf(w) = \{a,b\}\}$,

(c) $L_3 = \{w \in \Sigma^\omega : \{a,b\} \subseteq \inf(w)\}$,

(d) $L_4 = \{w \in \Sigma^\omega : \inf(w) = \{a,b,c\}\}$.

(e) ★ Does there exist a deterministic Büchi automaton accepting L_1? If there is then give it, otherwise give a proof of why it is not true.

Exercise 9.4
Prove or disprove:

(a) For every Büchi automaton A, there exists a Büchi automaton B with a single initial state and such that $L_\omega(A) = L_\omega(B)$;

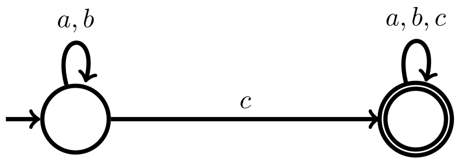
(b) For every Büchi automaton A, there exists a Büchi automaton B with a single accepting state and such that $L_\omega(A) = L_\omega(B)$;

(c) There exists a Büchi automaton recognizing the finite ω-language $\{w\}$ such that $w \in \{0,1,\ldots,9\}^\omega$ and w_i is the i^{th} decimal of $\sqrt{2}$.

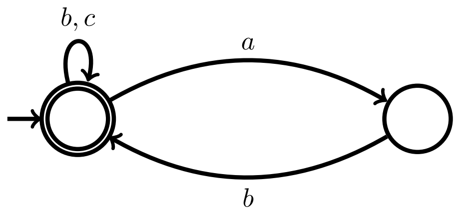
Solution 9.1
(1) $a^*(b^*(aa)^*)^\omega$.

(2) $a^*(b^*(\epsilon + aaa^*))^\omega$ or, one character shorter, $a^*(b^*(aaa^*)^*)^\omega$.

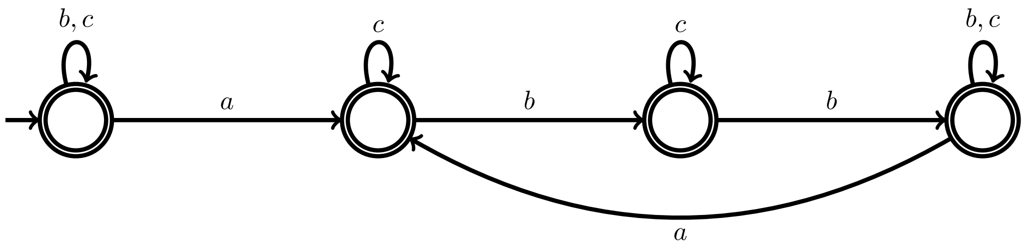
Solution 9.2
(a)



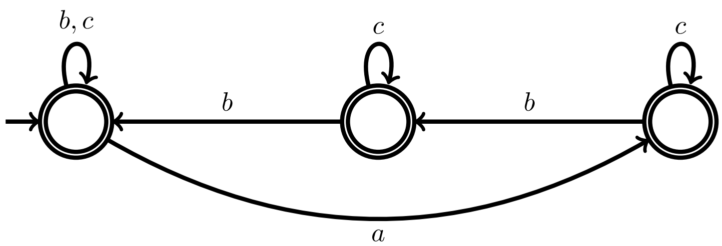
(b)



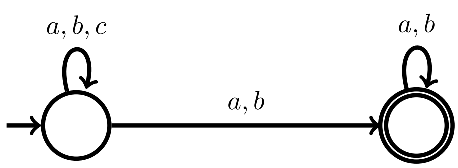
(c)



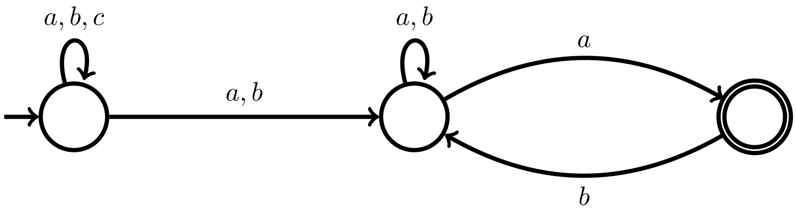
or simply,



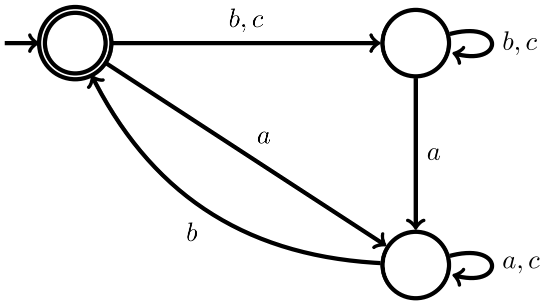
Solution 9.3
(a) $(a+b+c)^*(a+b)^\omega$, and



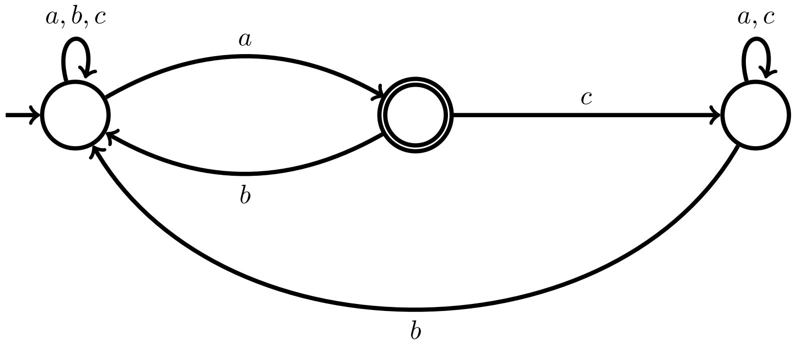
(b) $(a+b+c)^*(aa^*bb^*)^\omega$, and



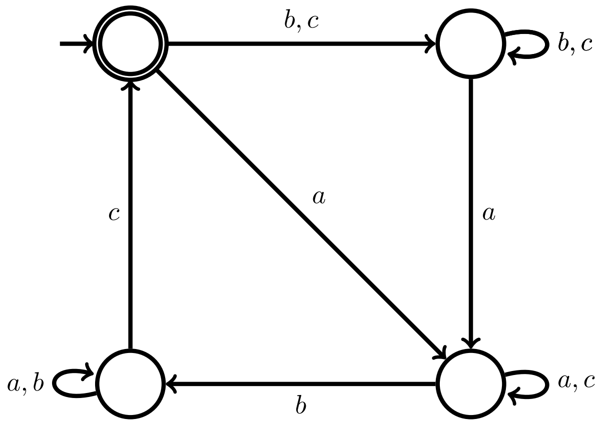
(c) $((b+c)^*a(a+c)^*b)^\omega$, and



or



(d) $((b+c)^*a(a+c)^*b(a+b)^*c)^\omega$, and



(e) ★ It is asked whether there exists a deterministic Büchi automaton accepting L_1. We show that it is *not* the case. For the sake of contradiction, suppose there exists a deterministic Büchi automaton $B = (Q, \Sigma, \delta, q_0, F)$ such that $L_\omega(B) = L_1$. Since $cb^\omega \in L_1$, B must visit F infinitely often when reading cb^ω. In particular, this implies the existence of $m_1 > 0$ and $q_1 \in F$ such that $q_0 \xrightarrow{cb^{m_1}} q_1$. Similarly, since $cb^{m_1}cb^\omega \in L_1$, there exist $m_2 > 0$ and $q_2 \in F$ such that $q_0 \xrightarrow{cb^{m_1}cb^{m_2}} q_2$. Since B is deterministic, we have $q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2$. By repeating this argument $|Q|$ times, we can construct $m_1, m_2, \ldots, m_{|Q|} > 0$ and $q_1, q_2, \ldots, q_{|Q|} \in F$ such that

$$q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2 \cdots \xrightarrow{cb^{m_{|Q|}}} q_{|Q|}.$$

By the pigeonhole principle, there exist $0 \leq i < j \leq |Q|$ such that $q_i = q_j$. Let

$$\begin{aligned} u &= cb^{m_1} cb^{m_2} \cdots cb^{m_i}, \\ v &= cb^{m_{i+1}} cb^{m_{i+2}} \cdots cb^{m_j}. \end{aligned}$$

We have $q_0 \xrightarrow{u} q_i \xrightarrow{v} q_i \xrightarrow{v} q_i \xrightarrow{v} \cdots$ which implies that $uv^\omega \in L_\omega(B)$. Also notice that c appears infinitely often in uv^ω, that is, $c \in \inf(uv^\omega)$. Therefore we have $uv^\omega \notin L_1 = L_\omega(B)$, which yields a contradiction. □

Solution 9.4

(a) True. The construction for NFAs still work for Büchi automata.

Let $B = (Q, \Sigma, \delta, Q_0, F)$ be a Büchi automaton. We add a state to Q which acts as the single initial state. More formally, we define $B' = (Q \cup \{q_{\text{init}}\}, \Sigma, \delta', \{q_{\text{init}}\}, F)$ where

$$\delta'(q, a) = \begin{cases} \bigcup_{q_0 \in Q_0} \delta(q_0, a) & \text{if } q = q_{\text{init}}, \\ \delta(q, a) & \text{otherwise.} \end{cases}$$

We have $L_\omega(B) = L_\omega(B')$, since there exists $q_0 \in Q_0$ such that

$$q_0 \xrightarrow{a_1}_B q_1 \xrightarrow{a_2}_B q_2 \xrightarrow{a_3}_B \cdots$$

if and only if

$$q_{\text{init}} \xrightarrow{a_1}_{B'} q_1 \xrightarrow{a_2}_{B'} q_2 \xrightarrow{a_3}_{B'} \cdots$$

(b) False. Let $L = \{a^\omega, b^\omega\}$. Suppose there exists a Büchi automaton $B = (Q, \{a, b\}, \delta, Q_0, F)$ such that $L_\omega(B) = L$ and $F = \{q\}$. Since $a^\omega \in L$, there exist $q_0 \in Q_0$, $m \geq 0$ and $n > 0$ such that

$$q_0 \xrightarrow{a^m} q \xrightarrow{a^n} q.$$

Similarly, since $b^\omega \in L$, there exist $q_0' \in Q_0$, $m' \geq 0$ and $n' > 0$ such that

$$q_0' \xrightarrow{b^{m'}} q \xrightarrow{b^{n'}} q.$$

This implies that

$$q_0 \xrightarrow{a^m} q \xrightarrow{b^{n'}} q \xrightarrow{b^{n'}} \cdots$$

Therefore, $a^m (b^{n'})^\omega \in L$, which is a contradiction. □

(c) False. Suppose there exists a Büchi automaton $B = (Q, \{0, 1, \ldots, 9\}, \delta, Q_0, F)$ such that $L_\omega(B) = \{w\}$. There exist $u \in \{0, 1, \ldots, 9\}^*$, $v \in \{0, 1, \ldots, 9\}^+$, $q_0 \in Q_0$ and $q \in F$ such that

$$q_0 \xrightarrow{u} q \xrightarrow{v} q.$$

Therefore, $uv^\omega \in L_\omega(B)$ which implies that $w = uv^\omega$. Since w represents the decimals of $\sqrt{2}$, we conclude that $\sqrt{2}$ is rational, which is a contradiction. □