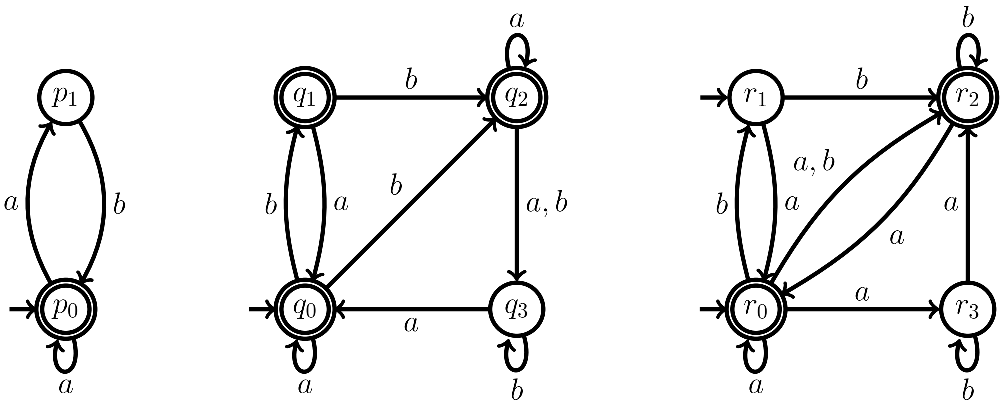
**Technische Universität München** **Winter term 22/23**
**I7**
**Prof. J. Esparza / A. R. Balasubramanian / M. Lazić**

# Automata and Formal Languages — Exercise Sheet 5

**Exercise 5.1**
Consider the following NFAs $A$, $B$ and $C$:




(a) Use algorithm *UnivNFA* to determine whether $L(B) = \{a, b\}^*$ and $L(C) = \{a, b\}^*$.

(b) For $D \in \{B, C\}$, if $L(D) \neq \{a, b\}^*$, use algorithm *InclNFA* to determine whether $L(A) \subseteq L(D)$.

**Exercise 5.2**
Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA. For any $S \subseteq Q$, a word $w \in \Sigma^*$ is said to be a *synchronizing word* for $S$ in $A$ if reading $w$ from any state of $S$ leads to a common state, i.e., if there exists $q \in Q$ such that **for every $\mathbf{p} \in \mathbf{S}$**, $p \xrightarrow{w} q$. We now define the *synchronizing word* problem defined as follows:

*Given:* DFA $A$ and a subset $S$ of the states of $A$
*Decide:* If there exists a synchronizing word for $S$ in $A$

(a) Given states $p, q \in Q$, design a polynomial time algorithm for testing if there is a synchronizing word for $\{p, q\}$ in $A$.

(b) Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Show that there is a synchronizing word for $Q$ in $A$ if and only if for every $p, q \in Q$, there is a synchronizing word for $\{p, q\}$ in $A$.

By (a) and (b), we can conclude that there is a polynomial time algorithm for the special case of the synchronizing word problem where the subset $S$ is the set of all states of $A$. However, for the general case, we have the following result.

(c) ★ Show that the synchronizing word problem is PSPACE-hard. You may assume that the following problem, called the *DFA intersection problem* is PSPACE-hard:

*Given:* DFAs $A_1, A_2, \ldots, A_n$ all over a common alphabet $\Sigma$
*Decide:* If there exists a word $w$ such that $w \in \bigcap_{1 \leq i \leq n} \mathcal{L}(A_i)$

**Exercise 5.3**
Let $\Sigma$ be a finite alphabet and let $L \subseteq \Sigma^*$ be a language accepted by an NFA $A$. Give an NFA-$\varepsilon$ for each of the following languages:

(a) $\sqrt{L} = \{w \in \Sigma^* \mid ww \in L\}$,

(b) ★ $\mathrm{Cyc}(L) = \{vu \in \Sigma^* \mid uv \in L\}$.

**Solution 5.1**

(a) The trace of the execution for NFA $B$ is as follows:

| **Iter.** | $\mathcal{Q}$ | $\mathcal{W}$ |
|---|---|---|
| 0 | $\emptyset$ | $\{\{q_0\}\}$ |
| 1 | $\{\{q_0\}\}$ | $\{\{q_1, q_2\}\}$ |
| 2 | $\{\{q_0\}, \{q_1, q_2\}\}$ | $\{\{q_2, q_3\}\}$ |
| 3 | $\{\{q_0\}, \{q_1, q_2\}, \{q_2, q_3\}\}$ | $\{\{q_3\}\}$ |

At the fourth iteration, the algorithm encounters state $\{q_3\}$ which is non final, and hence it returns *false*. Therefore, $L(B) \neq \{a, b\}^*$.

The trace of the execution for NFA $C$ is as follows:

| **Iter.** | $\mathcal{Q}$ | $\mathcal{W}$ |
|---|---|---|
| 0 | $\emptyset$ | $\{\{r_0, r_1\}\}$ |
| 1 | $\{\{r_0, r_1\}\}$ | $\{\{r_0, r_2, r_3\}, \{r_1, r_2\}\}$ |
| 2 | $\{\{r_0, r_1\}, \{r_0, r_2, r_3\}\}$ | $\{\{r_1, r_2\}\}$ |
| 3 | $\{\{r_0, r_1\}, \{r_0, r_2, r_3\}, \{r_1, r_2\}\}$ | $\{\{r_0\}, \{r_2\}\}$ |
| 3 | $\{\{r_0, r_1\}, \{r_0, r_2, r_3\}, \{r_1, r_2\}, \{r_0\}\}$ | $\{\{r_2\}\}$ |
| 3 | $\{\{r_0, r_1\}, \{r_0, r_2, r_3\}, \{r_1, r_2\}, \{r_0\}, \{r_2\}\}$ | $\emptyset$ |

At the fifth iteration, $\mathcal{W}$ becomes empty and hence the algorithm returns *true*. Therefore $L(C) = \{a, b\}^*$.

(b) The trace of the algorithm for $A$ and $B$ is as follows:

| **Iter.** | $\mathcal{Q}$ | $\mathcal{W}$ |
|---|---|---|
| 0 | $\emptyset$ | $\{[p_0, \{q_0\}]\}$ |
| 1 | $\{[p_0, \{q_0\}]\}$ | $\{[p_1, \{q_0\}]\}$ |
| 2 | $\{[p_0, \{q_0\}], [p_1, \{q_0\}]\}$ | $\{[p_0, \{q_1, q_2\}]\}$ |
| 3 | $\{[p_0, \{q_0\}], [p_1, \{q_0\}], [p_0, \{q_1, q_2\}]\}$ | $\emptyset$ |

At the third iteration, $\mathcal{W}$ becomes empty and hence the algorithm returns *true*. Therefore $L(A) \subseteq L(B)$.

**Solution 5.2**

(a) By definition, $w$ is a synchronizing word for $\{p, q\}$ in $A$ if and only if there is a state $r$ such that $r = \delta(p, w) = \delta(q, w)$. Consider the following algorithm: For every state $r \in Q$, we construct two DFAs $A_r^p = (Q, \Sigma, \delta, p, r)$ and $A_r^q = (Q, \Sigma, \delta, q, r)$. Notice that $w$ is a synchronizing word for $\{p, q\}$ in $A$ if and only if there exists a state $r$ such that $w \in \mathcal{L}(A_r^p) \cap \mathcal{L}(A_r^q)$. Hence, the polynomial time algorithm to test if there is a synchronizing word for $\{p, q\}$ in $A$ is as follows: For each $r \in Q$, construct the DFAs $A_r^p$ and $A_r^q$ and test if $\mathcal{L}(A_r^p) \cap \mathcal{L}(A_r^q) \neq \emptyset$ by means of the pairing construction and the emptiness check for DFAs. If for at least one state $r$, this test is true, then there is a synchronizing word for $\{p, q\}$ in $A$; otherwise, there is none.

To analyse the running time, note that we are doing at most $|Q|$ pairing constructions and emptiness checks, each of which takes polynomial time. Hence, the overall running time is also a polynomial in the size of the given input.

(b) $(\Rightarrow)$ : Suppose $w$ is a synchronizing word for $Q$ in $A$. Let $p, q \in Q$. By definition of a synchronizing word, $\delta(p, w) = \delta(q, w)$. Hence, $w$ is also a synchronizing word for $\{p, q\}$ in $A$.

$(\Leftarrow)$ : Suppose for every $p, q \in Q$, there is a synchronizing word $w_{p,q}$ for the subset $\{p, q\}$. We now construct a synchronizing word $w_S$ for every subset $S \subseteq Q$, by induction on $|S|$, the size of $S$.

For the base case, note that if $|S| = 1$, then $\epsilon$ is a synchronizing word for $S$. Assume that we have shown that whenever $|S| \leq i$ for some number $i \geq 1$, there is a synchronizing word for $S$. Suppose $S$ is a subset such that $|S| = i + 1$. Hence, $|S| \geq 2$ and let $S = \{p_1, p_2, \dots, p_{i+1}\}$. By assumption, there is a synchronizing word $w_{p_1,p_2}$ for the subset $\{p_1, p_2\}$. Let $S' = \{\delta(p_1, w_{p_1,p_2}), \delta(p_2, w_{p_1,p_2}), \dots, \delta(p_{i+1}, w_{p_1,p_2})\}$. Since $w_{p_1,p_2}$ is a synchronizing word for $\{p_1, p_2\}$, it follows that $|S'| \leq i$. By induction hypothesis, there is a synchronizing word $w_{S'}$ for the subset $S'$. It is then easy to see that the word $w_{p_1,p_2} w_{S'}$ is a synchronizing word for $S$ in $A$. Hence, the induction step is complete.

It then follows that there is a synchronizing word for the set $Q$ in $A$.

(c) We give a polynomial-time reduction from the DFA intersection problem to the synchronizing word problem, which will prove that the latter is PSPACE-hard. Let $A_1, A_2, \dots, A_n$ be $n$ DFAs all over a commmon alphabet $\Sigma$ such that each $A_i = (Q_i, \Sigma, \delta_i, q_0^i, F_i)$. In polynomial time, we have to construct a DFA $B$ and a subset $S$ of the states of $B$ so that

$$S \text{ has a synchronizing word in } B \text{ if and only if } \bigcap_{1 \leq i \leq n} \mathcal{L}(A_i) \neq \emptyset$$

Let us construct $B = (Q_B, \Sigma_B, \delta_B, q_0^B, F_B)$ and $S$ as follows.

- The set $Q_B$ will consist all the states of all the $A_i$'s and in addition, it will have two new states $p$ and $t$. More formally, $Q = \bigcup_{1 \leq i \leq n} Q_i \cup \{p, t\}$ where $p$ and $t$ are two new states.
- The alphabet $\Sigma_B$ will be $\Sigma \cup \{\#\}$ where $\#$ is a fresh letter not present in $\Sigma$.
- The transition function $\delta_B$ will behave in the following way:
  - If $q \in Q_i$ for some $i$ and $a \in \Sigma$, then $\delta_B(q, a) = \delta_i(q, a)$. Intuitively, if $q$ is a state of some $A_i$ and $a$ is not $\#$, then the transition function behaves in exactly the same way as $\delta_i$.
  - If $q \in F_i$ for some $i$, then $\delta_B(q, \#) = p$. Intuitively, upon reading a $\#$ from some accepting state of some $A_i$, we move to $p$.
  - If $q \in Q_i \setminus F_i$ for some $i$, then $\delta_B(q, \#) = t$. Intuitively, upon reading a $\#$ from some rejecting state of $A_i$, we move to $t$.
  - If $q \in \{p, t\}$ and $a \in \Sigma_B$, then $\delta_B(q, a) = q$. Intuitively, the states $p$ and $t$ have a self-loop corresponding to any letter.
- We set $q_0^B$ to be $p$ and $F_B$ to be $\{p\}$.
- Finally we set $S$ to be the subset of states given by $\{q_0^1, q_0^2, \dots, q_0^n, p\}$.

Suppose $w \in \bigcap_{1 \leq i \leq n} \mathcal{L}(A_i)$. By construction, it then follows that $w\#$ is a synchronizing word for $S$ in $B$.

Suppose $w$ is a synchronizing word for $S$ in $B$. By definition of $w$ and by construction of $B$, it follows that

$$\delta_B(q_0^1, w) = \delta_B(q_0^2, w) = \cdots = \delta_B(q_0^n, w) = \delta_B(p, w) = p$$

Notice that to move from the state $q_0^1$ to $p$, it is necessary to read a $\#$ at some point. Hence, $w$ must contain an occurrence of $\#$. Split $w$ as $w'\#w''$ so that $w'$ has no occurrences of $\#$. For each $i$, let $q_i = \delta_B(q_0^i, w')$. By construction of $B$, it follows that for each $i$, $q_i \in Q_i$. Suppose for some $i$, $q_i \notin F_i$. It then follows that $\delta_B(q_i, \#w'') = t$, which contradicts the fact that $\delta_B(q_0^i, w\#w'') = p$. Hence, $q_i \in F_i$ for every $i$ and this implies that $w'$ is a word which is accepted by all of the $A_i$'s.

**Solution 5.3**

Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NFA that accepts $L$. Without loss of generality, we can assume that $Q_0 = \{q_0\}$ and $F = \{q_f\}$ for some states $q_0$ and $q_f$.

(a) To begin with we have the following observation:

$$w \in \sqrt{L} \text{ if and only if there exists a state } p \in Q \text{ such that } p \in \delta(q_0, w) \text{ and } q_f \in \delta(p, w).$$

With this observation in mind, let us do the following construction: For every state $p \in Q$, construct two NFAs $A_p^1, A_p^2$ defined as $A_p^1 = (Q, \Sigma, \delta, q_0, p)$ and $A_p^2 = (Q, \Sigma, \delta, p, q_f)$. Notice that we can now rephrase the above observation as:

$$w \in \sqrt{L} \text{ if and only if there exists a state } p \in Q \text{ such that } w \in \mathcal{L}(A_p^1) \cap \mathcal{L}(A_p^2).$$

Let $B$ be any NFA for the language $\cup_{p \in Q} \mathcal{L}(A_p^1) \cap \mathcal{L}(A_p^2)$. By the above observation, it follows that $B$ recognizes $\sqrt{L}$. Note that $B$ can be obtained by pairing operations on the NFAs from the set $\{A_p^i : p \in Q,\ i \in \{1,2\}\}$ and each element in this set can be easily constructed from $A$. It follows then that we can explicitly construct $B$ from $A$.

(b) Once again we begin with an observation:

> $w = w_1 w_2 \dots w_n \in \text{Cyc}(L)$ if and only if there exists $1 \leq i \leq n$ and $p \in Q$ such that $q_f \in \delta(p, w_1 w_2 \dots w_i)$ and $p \in \delta(q_0, w_{i+1} w_{i+2} \dots w_n)$.

Indeed, suppose for some word $w$, such an $i$ and $p$ exist. Then, notice that if we set $v = w_1 w_2 \dots w_i$ and $u = w_{i+1} \dots w_n$, then $uv \in L$ and so $w = vu \in \text{Cyc}(L)$. On the other hand if $w \in \text{Cyc}(L)$, then there is a partition of $w$ into some $v = w_1 w_2 \dots w_i$ and $u = w_{i+1} \dots w_n$ such that $uv \in L$. Since $uv \in L$, there must be an accepting run of $uv$ in $A$. Let $p$ be the state reached after reading $u$ along this run. It follows then that $p \in \delta(q_0, w_{i+1} \dots w_n)$ and $q_f \in \delta(p, w_1 w_2 \dots w_i)$.

With this observation, we can do the following: For every state $p \in Q$, construct the two NFAs $A_p^1$ and $A_p^2$ as defined in the subproblem a). Now, notice that

> $w \in \text{Cyc}(L)$ if and only if there exists $p \in Q$ such that $w \in \mathcal{L}(A_p^2)\mathcal{L}(A_p^1)$, i.e., $w$ is in the concatenation of the languages of $A_p^2$ and $A_p^1$ for some $p$.

Let $B$ be any NFA for the regular language $\cup_{p \in Q} \mathcal{L}(A_p^2)\mathcal{L}(A_p^1)$. By the above observation, it follows that $B$ recognizes $\text{Cyc}(L)$. Note that given the NFAs $A_p^2$ and $A_p^1$, we can obtain an NFA-$\epsilon$ for their concatenation by simply adding an $\epsilon$ transition from the final state of $A_p^2$ to the initial state of $A_p^1$. By using additional pairing operations, we can explicitly construct $B$ from $A$.