Technische Universität München **Winter term 21/22**
I7
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Automata and Formal Languages — Exercise Sheet 5

Exercise 5.1
Consider the following languages over alphabet $\Sigma = \{a, b\}$:

- L_1 is the set of all words where a occurs only at odd positions;
- L_2 is the set of all words with an even number of a's;
- L_3 is the set of all words where between any two occurrences of b's there is at least one a;
- L_4 is the set of all words of odd length.

Construct an NFA for the language

$$(L_1 \setminus L_2) \cup \overline{(L_3 \triangle L_4)},$$

where $L \triangle L'$ denotes the symmetric difference of L and L', i.e. $(L \setminus L') \cup (L' \setminus L)$, while sticking to the following rules:

- Start from DFAs for $L_1, \ldots, L_4$;
- Any further automaton must be constructed from already existing automata via an algorithm introduced in the lecture, e.g. *Comp*, *BinOp*, *UnionNFA*, *NFAtoDFA*, etc.

Exercise 5.2
Let Σ be a finite alphabet. For every $u, v \in \Sigma^*$, we say that $u \preceq v$ if and only if u can be obtained by deleting zero or more letters of v. For example, $abc \preceq abca$, $abc \preceq acbac$, $abc \preceq abc$, $\varepsilon \preceq abc$ and $aab \not\preceq acbac$.

Let $L \subseteq \Sigma^*$ be a language accepted by an NFA A. Give an NFA-ε for each of the following languages:

(a) $\downarrow L = \{w \in \Sigma^* \mid w \preceq w' \text{ for some } w' \in L\}$,

(b) $\uparrow L = \{w \in \Sigma^* \mid w' \preceq w \text{ for some } w' \in L\}$,

(c) $\sqrt{L} = \{w \in \Sigma^* \mid ww \in L\}$,

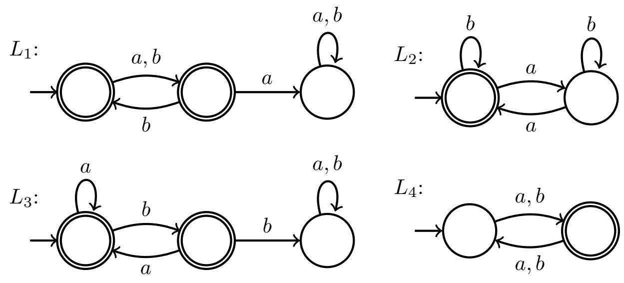
(d) $\bigstar$ $\mathrm{Cyc}(L) = \{vu \in \Sigma^* \mid uv \in L\}$.

Exercise 5.3
Let $L \neq \{\epsilon\}$ be an arbitrary non-empty language over a 1-letter alphabet. Prove that there exists words $v_1, v_2, \ldots, v_n, w$ such that $L^* = (v_1 + v_2 + \cdots + v_n)w^*$.

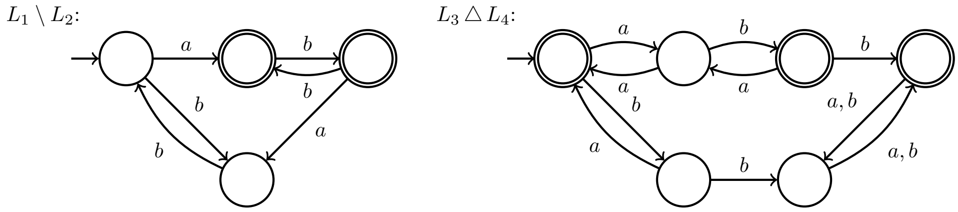
(**Hint:** Consider the shortest non-empty word $w \in L$. If $L^* = w^*$, then we are done. Otherwise, pick the shortest word $v_1 \in L^* \setminus w^*$. If $L^* = v_1 w^*$, then we are done. Otherwise, pick the shortest word $v_2 \in L^* \setminus v_1 w^*$ and so on).

Solution 5.1

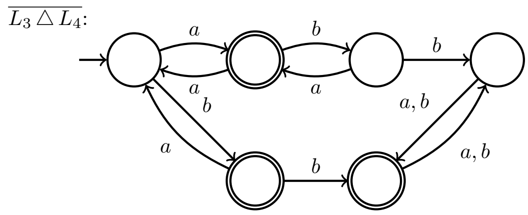
We start from the following deterministic automata:



By applying *BinOp* (and omitting the trap state on $L_1 \setminus L_2$), we obtain:



By using *Comp* on the rightmost automaton, we obtain:



By considering the NFA for $L_1 \setminus L_2$ and the above NFA as a single automaton, we obtain an NFA for $(L_1 \setminus L_2) \cup \overline{(L_3 \triangle L_4)}$.

Solution 5.2

Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NFA that accepts L.

(a) We add a nε-transition "parallel" to every transition of A. This simulates the deletion of letters from words of L. More formally, let $B = (Q, \Sigma, \delta', Q_0, F)$ be such that, for every $q \in Q$ and $a \in \Sigma \cup \{\varepsilon\}$,

$$\delta'(q,a) = \begin{cases} \delta(q,a) & \text{if } a \in \Sigma, \\ \{q \in Q : q \in \delta(q,b) \text{ for some } b \in \Sigma\} & \text{if } a = \varepsilon. \end{cases}$$

(b) For every state of Q, we add self-loops for each letter of Σ. This corresponds to the insertion of letters in words of L. More formally, let $B = (Q, \Sigma, \delta', Q_0, F)$ be such that $\delta'(q,a) = \delta(q,a) \cup \{q\}$ for every $q \in Q$ and $a \in \Sigma$.

(c) Intuitively, we construct an automaton B that guesses an intermediate state p and then reads w simultaneously from an initial state q_0 and from p. The automaton accepts if it simultaneously reaches p and and an accepting state q_F. More formally, let $B = (Q', \Sigma, \delta', Q'_0, F')$ be such that

$$\begin{aligned} Q' &= Q \times Q \times Q, \\ Q'_0 &= \{(p,q,p) : p \in Q, q \in Q_0\}, \\ F' &= \{(p,p,q) : p \in Q, q \in F\}, \end{aligned}$$

and, for every $p, q, r \in Q$ and $a \in \Sigma$,

$$\delta'((p, q, r), a) = \{(p, q', r') : q' \in \delta(q, a), r' \in \delta(r, a)\}.$$

(d) Intuitively, we construct an automaton B that guesses a state p and reads a prefix v of the input word until it reaches a final state. Then, B moves non deterministically to an initial state from which it reads the remainder u of the input word, and it accepts if it reaches p. More formally, let $B = (Q', \Sigma, \delta', Q'_0, F')$ be such that

$$\begin{aligned} Q' &= Q \times \{0, 1\} \times Q, \\ Q'_0 &= \{(p, 0, p) \mid p \in Q\}, \\ F' &= \{(p, 1, p) \mid p \in Q\}, \end{aligned}$$

and, for every $p, q \in Q$ and $a \in \Sigma \cup \{\varepsilon\}$,

$$\delta'((p, b, q), a) = \begin{cases} \{(p, b, q') : q' \in \delta(q, a)\} & \text{if } a \in \Sigma, \\ \{(p, 1, q') : q' \in Q_0\} & \text{if } a = \varepsilon, b = 0 \text{ and } q \in F, \\ \emptyset & \text{otherwise.} \end{cases}$$

Solution 5.3

Without loss of generality, we can assume that the alphabet is $\{a\}$. As the hint suggests, we first consider the shortest non-empty word $w \in L$. If $L^* = w^*$, then we are done. Otherwise, there must be a shortest word $v_1 \in L^* \setminus w^*$. If $L^* = v_1 w^*$, then we are done again. Otherwise, there must be a shortest word $v_2 \in L^* \setminus v_1 w^*$ and so on.

We claim that in atmost $p = |w|$ steps, this process will terminate and we will find words $v_1, \ldots, v_n, w$ that satisfy the required claim. Indeed, suppose this process does not terminate in atmost p steps and so we have constructed words $v_1, v_2, \ldots, v_{p+1}$. By the pigeonhole principle, there exists $1 \leq i < j \leq p + 1$ such that $|v_i| \equiv |v_j| \pmod{p}$. Notice that $|v_i| \neq |v_j|$ as otherwise $v_i = v_j$, because both of them are words over a singleton alphabet. Hence we have two cases.

Suppose $|v_i| < |v_j|$. Since $|v_i| \equiv |v_j|$ (mod p), there must be a $k > 0$ such that $|v_j| = |v_i| + k \cdot p$. Hence, $v_j = a^{|v_j|} = a^{|v_i|+k \cdot p} = v_i w^k \in v_i w^*$, contradicting the way v_j was picked.

Suppose $|v_j| < |v_i|$. Since $|v_i| \equiv |v_j|$ (mod p), there must be a $k > 0$ such that $|v_i| = |v_j| + k \cdot p$. Hence, $v_i = a^{|v_i|} = a^{|v_j|+k \cdot p} = v_j w^k \in v_j w^*$. If $v_j \in (v_1 + v_2 + \cdots + v_{i-1})w^*$, this would then mean that $v_i \in (v_1 + v_2 + \cdots + v_{i-1})w^*$ as well, contradicting the way v_i was picked. Otherwise, $v_j \notin (v_1 + v_2 + \cdots + v_{i-1})w^*$, but then $|v_j| < |v_i|$, which also contradicts the choice of v_i. It follows that in either case, we arrive at a contradiction.

Hence, the process terminates in atmost p steps. Since the process terminates, it means that we have found $v_1, \ldots, v_n, w$ satisfying the property that $L^* = (v_1 + v_2 + \cdots + v_n)w^*$.

Remark: A previous version of this question which also required that the words $v_1, \ldots, v_n, w$ belong to L, was wrong. This has now been corrected.