Technische Universität München **Winter term 22/23**
I7
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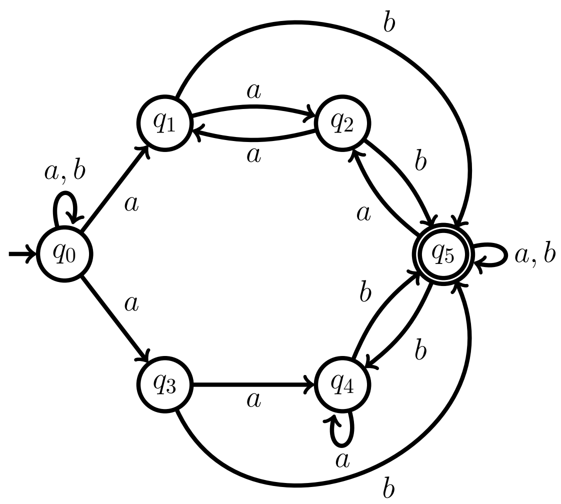
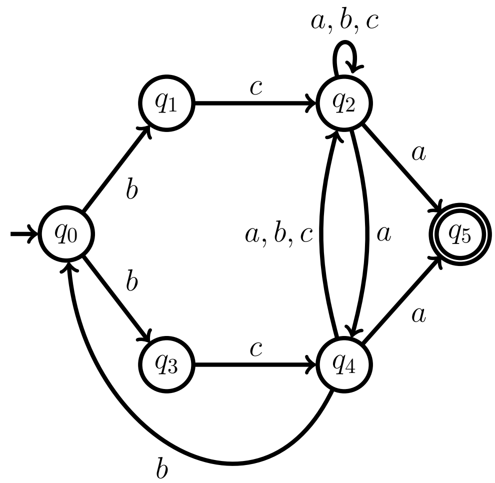
Automata and Formal Languages — Exercise Sheet 4

Exercise 4.1
Let A and B be respectively the following NFAs:





(a) Compute the coarsest stable refinements (CSR) of A and B.

(b) Construct the quotients of A and B with respect to their CSRs.

(c) Show that

$$L(A) = \{w \in \{a,b\}^* : w \text{ contains an occurrence of the subword } ab\}$$
$$L(B) = \{w \in \{a,b,c\}^* : w \text{ starts with } bc \text{ and ends with } a\}$$

(d) Are the automata obtained in (b) minimal?

Exercise 4.2
Let $\Sigma = \{a, b\}$. For any $n \in \mathbb{N}$, let $L_n := \{ww^R : w \in \Sigma^n\}$, where w^R is the reverse of w, e.g. $(abc)^R = cba$. In Exercise 2.3, we have shown that every NFA (and hence also every DFA) recognizing L_n must have at least 2^n states. We refine this bound here for DFAs.

(a) Construct A_2, the minimal DFA for L_2.

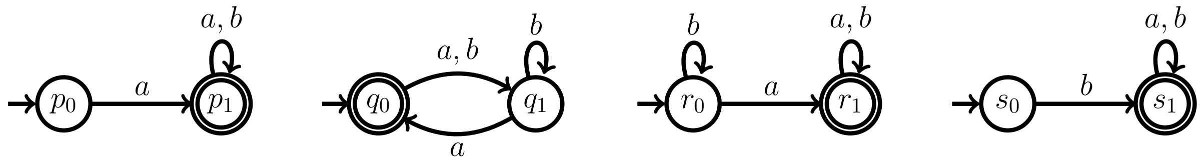
(b) What are the residuals of L_2? Assign them to the states of the DFA you gave for (a).

(c) Give a construction for a DFA that accepts L_n.

(d) How many states does the minimal DFA for L_n contain, for $n \geq 2$?

Exercise 4.3

Consider the following DFAs A, B, C and D:



(a) Use pairings to decide *algorithmically* whether $L(A) \cap L(B) \subseteq L(C)$.

(b) Use pairings to decide *algorithmically* whether $L(D) \subseteq L(A) \cap L(B)$.

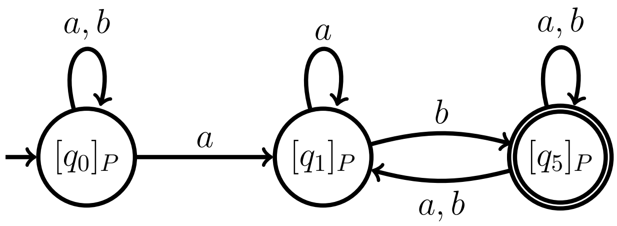
Solution 4.1

A) (a)

Iter.	Block to split	Splitter	New partition
0	—	—	$\{q_0, q_1, q_2, q_3, q_4\}, \{q_5\}$
1	$\{q_0, q_1, q_2, q_3, q_4\}$	$(b, \{q_5\})$	$\{q_0\}, \{q_1, q_2, q_3, q_4\}, \{q_5\}$
2	none, partition is stable	—	—

The CSR is $P = \{\{q_0\}, \{q_1, q_2, q_3, q_4\}, \{q_5\}\}$.

(b)



(c) The automaton A and the automaton obtained from (b) accept the same language. Notice that in the automaton from (b), there is an accepting run for a word w which visits the final state exactly once if and only if $w \in \Sigma^* ab$. Since there are self-loops at the final state for both a and b, it follows that the language of this automaton is $\Sigma^* ab \Sigma^*$.

(d) Yes. By (c), the language accepted by A is $\Sigma^* ab \Sigma^*$. An NFA with one state can only accept $\emptyset, \{\varepsilon\}, a^*, b^*$ and $\{a, b\}^*$. Suppose there exists an NFA $A' = (\{q_0, q_1\}, \{a, b\}, \delta, Q_0, F)$ accepting $L(A)$. Without loss of generality, we may assume that q_0 is initial. A' must respect the following properties:

- $q_0 \notin F$, since $\varepsilon \notin L(A)$,
- $q_1 \in F$, since $L(A) \neq \emptyset$,
- $q_1 \notin Q_0$, since $\varepsilon \notin L(A)$,
- $\delta(q_0, a)$ is non-empty, otherwise it is impossible to accept ab. Further, $q_1 \notin \delta(q_0, a)$, otherwise it is possible to accept a. Hence, $\delta(q_0, a) = \{q_0\}$.
- $q_1 \in \delta(q_0, b)$, otherwise it is impossible to accept ab.

This implies that A' accepts b, yet $b \notin L(A)$. Therefore, no NFA with two states can accept $L(A)$. $\square$

B) (a)

Iter.	Block to split	Splitter	New partition
0	—	—	$\{q_0, q_1, q_2, q_3, q_4\}, \{q_5\}$
1	$\{q_0, q_1, q_2, q_3, q_4\}$	$(a, \{q_5\})$	$\{q_0, q_1, q_3\}, \{q_2, q_4\}, \{q_5\}$
2	$\{q_2, q_4\}$	$(b, \{q_0, q_1, q_3\})$	$\{q_0, q_1, q_3\}, \{q_2\}, \{q_4\}, \{q_5\}$
3	$\{q_0, q_1, q_3\}$	$(c, \{q_4\})$	$\{q_0, q_1\}, \{q_3\}, \{q_2\}, \{q_4\}, \{q_5\}$
4	$\{q_0, q_1\}$	$(c, \{q_2\})$	$\{q_0\}, \{q_1\}, \{q_3\}, q_2, \{q_4\}, \{q_5\}$

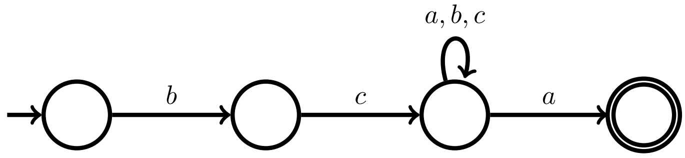
The CSR is $P = \{\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}, \{q_5\}\}$.

(b) The automaton remains unchanged.

(c) $\supseteq$) Suppose w starts with bc and ends with a. If $w = w_1 w_2 \dots w_n$, then $q_0, q_1, \underbrace{\dots, q_2, \dots,}_{n-3 \text{ times}} q_5$ is a valid accepting run for w.

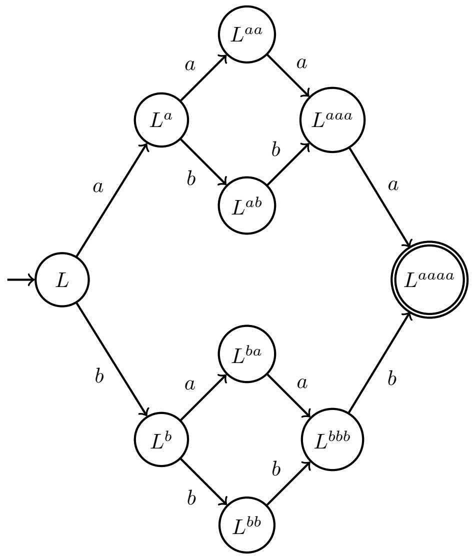
$\subseteq$) Let $w \in L(B)$. Note that every outgoing edge from q_0 is labelled by a b and goes to either q_1 or q_3 and every outgoing edge from both q_1 and q_3 is labelled by a c. It follows that any path from q_0 to q_5 must involve reading a bc at the beginning. Further, all the incoming edges to q_5 are labelled by an a. It follows that any path from q_0 to q_5 must involve reading an a at the end. Since $w \in L(B)$, it then follows that w must begin with bc and end with a. $\square$

(d) No. The following NFA with four states accepts the same language.



Solution 4.2

(a) The trap state is omitted for the sake of readability:



(b) We have $L_2 = \{aaaa, abba, baab, bbbb\}$. We compute the residuals L^w for all words w by increasing length of w.

- $|w| = 0$: $L^\varepsilon = \{aaaa, abba, baab, bbbb\}$.
- $|w| = 1$: $L^a = \{aaa, bba\}$ and $L^b = \{aab, bbb\}$.
- $|w| = 2$: $L^{aa} = \{aa\}$, $L^{ab} = \{ba\}$, $L^{ba} = \{ab\}$ and $L^{bb} = \{bb\}$.
- $|w| = 3$: $L^{aaa} = \{a\} = L^{abb}$, and $L^{baa} = \{b\} = L^{bbb}$.
- $|w| \geq 4$: $L^w = \begin{cases} \{\varepsilon\} & \text{if } w \in L_k, \\ \emptyset & \text{otherwise.} \end{cases}$

(c) Notice that L_{k+1} is simply $aL_ka + bL_kb$ for any $k \geq 2$. Using this observation, we generalize the construction given in (a) for $k = 2$, by induction on k. The base case of $k = 2$ has been done already. Suppose we have already constructed $A_k = (Q_k, \{a, b, \}, \delta_k, q_0^k, q_f^k)$ with the property that it has exactly one initial state, one final state and one trap state $trap_k$ (Note that A_2 satisfies this property). We now construct $A_{k+1} = (Q_{k+1}, \{a, b, \}, \delta_{k+1}, q_0^{k+1}, q_f^{k+1})$ as follows:

The set of states Q_{k+1} is taken to be $\{q_0^{k+1}, q_f^{k+1}, trap_{k+1}\} \cup ((Q_k \setminus \{trap_k\}) \times \{1, 2\})$, where $q_0^{k+1}, q_f^{k+1}, trap_{k+1}$ are three fresh states. Intuitively we add a fresh initial state, a fresh final state, a fresh trap state and take two *copies* of the states of A_k while removing $trap_k$.

The transition function δ_{k+1} is defined as follows:

- $\delta_{k+1}(q_0^{k+1}, a) = (q_0^k, 1)$ and $\delta_{k+1}(q_0^{k+1}, b) = (q_0^k, 2)$. Intuitively, upon reading an a (resp. b) from the initial state of A_{k+1}, we move to the initial state of the first (resp. second) copy of A_k.
- $\delta_{k+1}(q_f^k, a) = q_f^{k+1}$ and $\delta_{k+1}(q_f^k, b) = q_f^{k+1}$. Intuitively, upon reading an a (resp. b) from the final state of the first (resp. second) copy of A_{k+1}, we move to the final state of of A_{k+1}.
- $\delta_{k+1}((q, i), a) = p$ where $p = (\delta_k(q, a), i)$ if $\delta_k(q, a) \neq trap_k$ and otherwise $p = trap_{k+1}$. Intuitively, within a copy of A_k, we follow the transitions of A_k and stay within that copy itself if the state that we are supposed to go to is not the trap state of A_k. Otherwise, instead of going to the trap state of A_k, we go to the trap state of A_{k+1}.

Assuming that A_k recognizes L_k, we can then show that A_{k+1} recognizes L_{k+1}. By induction, this will show that our construction is correct.

(d) Note that if $f(k)$ is the number of states that A_k has, (where A_k is the DFA defined in the previous subproblem), then $f(2) = 11$ and $f(k+1) = 2(f(k)-1)+3 = 2f(k)+1$. Solving this, we get $f(k) = 3 \cdot 2^k - 1$. We claim that A_k is a minimal DFA, by induction on k. The base case of $k = 2$ is already done. For the induction step, suppose p, q are two distinct states of A_{k+1}. We will show that $L_{A_{k+1}}(p) \neq L_{A_{k+1}}(q)$.

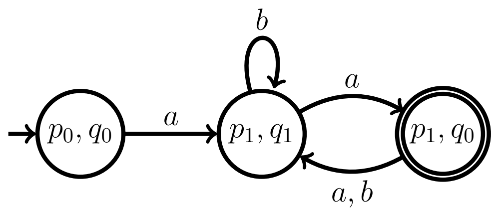
Notice that the initial state q_0^{k+1} recognizes only strings of length $2k+2$ and the final state q_f^{k+1} recognizes only ϵ, whereas the other states of A_{k+1} do not recognize any of these strings. This implies that the languages of the initial and the final states are different from the rest. Similarly, the language of the trap state is also different from the rest.

Hence, we can assume that $p = (p', i)$ and $q = (q', j)$ for some $p', q' \in Q_k$ and some $i, j \in \{1, 2\}$. If $i \neq j$, then p and q belong to different copies of A_k. Let $i = 1$ and $j = 2$. Notice that $L_{A_{k+1}}(p) = L_{A_k}(p')a$ and $L_{A_{k+1}}(q) = L_{A_k}(q')b$. Hence $L_{A_{k+1}}(p) \neq L_{A_{k+1}}(q)$.

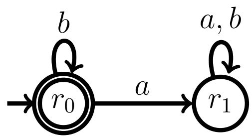
The only case left is when $i = j$. In this case notice that $L_{A_{k+1}}(p) = L_{A_k}(p')c$ and $L_{A_{k+1}}(q) = L_{A_k}(q')c$ where c is either a or b, depending on whether i is 1 or 2. By induction hypothesis, A_k is the minimal DFA for L_k and so $L_{A_k}(p')$ and $L_{A_k}(q')$ are different. Hence $L_{A_{k+1}}(p) \neq L_{A_{k+1}}(q)$, thereby concluding the proof.

Solution 4.3

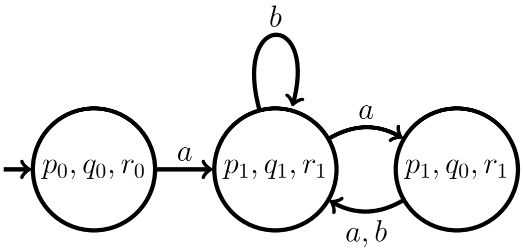
(a) We first build the pairing accepting $L(A) \cap L(B)$. Note that it is not necessary to explore the implicit trap states of A and B as they cannot lead to final states in the pairing. We obtain:



Now, we build the pairing accepting $(L(A) \cap L(B)) \setminus L(C)$, or equivalently $(L(A) \cap L(B)) \cap \overline{L(C)}$, from the above automaton and C. Recall that the complement of C is the following automaton:

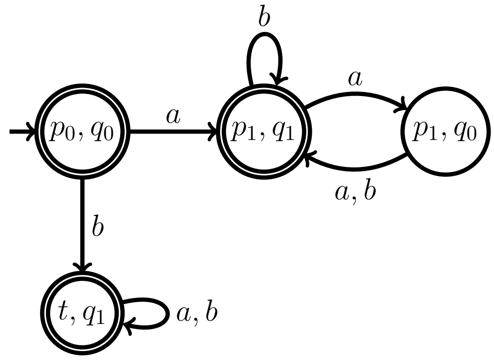


Once again, it is not necessary to explore the implicit trap states of the automaton for $L(A) \cap L(B)$. The following automaton is the pairing accepting $(L(A) \cap L(B)) \cap \overline{L(C)}$:

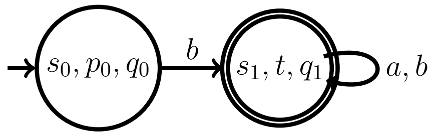


Since the above automaton does not contain final states, its language is empty and hence $L(A) \cap L(B) \subseteq L(C)$.

(b) This time we want to check whether $L(D) \setminus (L(A) \cap L(B))$ is empty. That is, we need to construct the pairing $L(D) \cap \overline{(L(A) \cap L(B))}$. Thus, it is not necessary to explore the implicit trap states of the automaton D, but it is necessary for A and B, as their trap states may be part of final states in the pairing. First we obtain the automaton accepting $\overline{(L(A) \cap L(B))}$:



Now, we build the pairing accepting $L(D) \cap \overline{(L(A) \cap L(B))}$. We obtain:



Since the above automaton contains a final state, it means that there is a word in the language $L(D) \setminus (L(A) \cap L(B))$, that is, there is a word accepted by D, but not by A and B. For example, any word starting with a letter b. Therefore, it is not true that $L(D) \subseteq L(A) \cap L(B)$.