Technische Universität München **Winter term 22/23**
I7
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Automata and Formal Languages — Exercise Sheet 1

You can find additional exercises in the Automata Tutor tool, for which the course name and password are available on the Moodle website for "Lecture and Exercises for Automata and Formal Languages (IN2041)". If you are enrolled for the course "Exercise - Automata and Formal Languages (IN2041)" in TUM online, you automatically have access to the Moodle website.

Exercise 1.1
Give a regular expression and a NFA for the language of all words over $\Sigma = \{a, b\}$ ...

1. ... beginning and ending with a.
2. ... such that the third letter from the right is a b.
3. ... that can be obtained from $babbab$ by deleting letters.
4. ... with no occurrences of the subword bba.
5. ... with at most one occurrence of the subword bba.

Exercise 1.2
Let A, B and C be three languages.

1. Prove that if $A \subseteq BC$ then $A^* \subseteq (B^* + C^*)^*$. Is the converse true?
2. Prove that the languages of $((a + ba)^* + b^*)^*$ and $(a + b)^*$ are the same.

Exercise 1.3
Consider the language $L \subseteq \{a, b\}^*$ given by the regular expression $a^*b(ba)^*a$.

1. Give an NFA that accepts L.
2. Give a DFA that accepts L.

Exercise 1.4
Let $\Sigma = \{a, b\}$ and let $\Sigma^* = (a + b)^*$. Suppose $w = a_1a_2 \dots a_n$ where each $a_i \in \Sigma$. Then the *upward closure* of a word w is defined as the set

$$\uparrow w = \{u_1a_1u_2a_2 \dots u_na_nu_{n+1} : u_1, u_2, \dots, u_{n+1} \in \Sigma^*\}$$

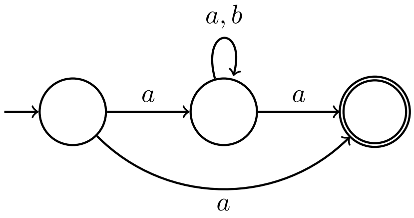
The *upward closure* of a language L is defined as the set $\uparrow L = \cup_{w \in L} \uparrow w$.

1. Give an algorithm that takes as input a regular expression r and outputs a regular expression $\uparrow r$ such that $\mathcal{L}(\uparrow r) = \uparrow (\mathcal{L}(r))$.
2. Give an algorithm that takes as input an NFA A and outputs an NFA B with exactly the same number of states as A such that $\mathcal{L}(B) = \uparrow \mathcal{L}(A)$.

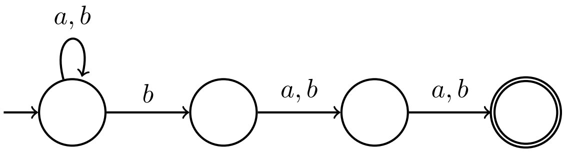
Solution 1.1

We write Σ for $(a+b)$ and Σ^* for $(a+b)^*$.

1. $a + (a\Sigma^* a)$

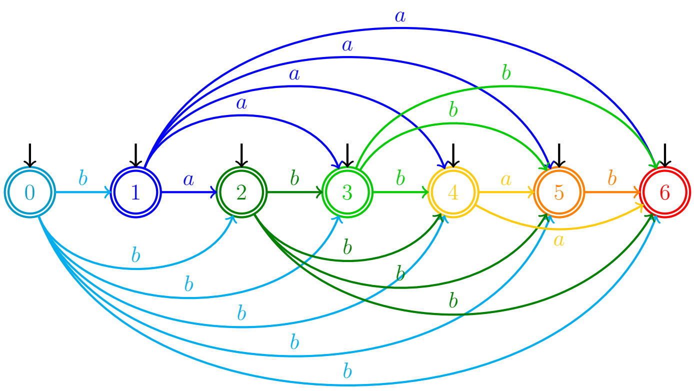


2. $\Sigma^* b \Sigma\Sigma$

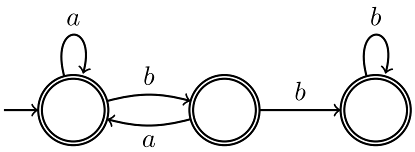


3. $(b+\epsilon)(a+\epsilon)(b+\epsilon)(b+\epsilon)(a+\epsilon)(b+\epsilon)$

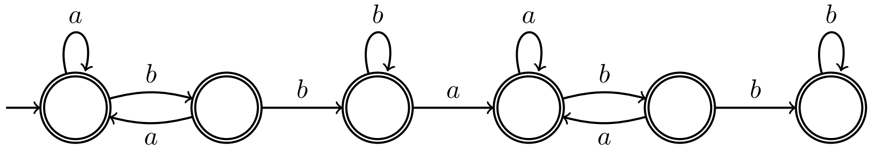
   One possible NFA for the language is the following. Note that every state of this NFA is initial and accepting. There are 7 states, labelled by 0, 1, 2, 3, 4, 5 and 6. From 0, upon reading a b, we can go to any state strictly bigger than 0; From 1, upon reading an a, we can go to any state strictly bigger than 1, and so on.



4. $(a+ba)^*b^*$



5. $((a+ba)^*b^*) + ((a+ba)^*b^*(bba)(a+ba)^*b^*)$



Solution 1.2

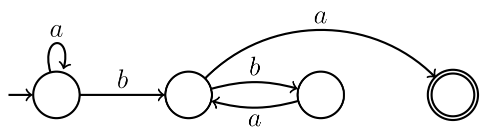
1. Suppose $A \subseteq BC$. First, we show that $A^* \subseteq (BC)^*$. Indeed, if $w \in A^*$, then w can be decomposed as $w_1 w_2 \ldots w_n$ for some number n such that each $w_i \in A$. Since $A \subseteq BC$, it follows that each $w_i \in BC$ and so $w \in (BC)^*$.

   Now, we show that $(BC)^* \subseteq (B^* + C^*)^*$. If $w \in (BC)^*$ then w can be decomposed as $w_1 w_2 \ldots w_n$ for some number n such that each $w_i \in BC$. Since each $w_i \in BC$, it follows that each w_i can be further decomposed as $u_i v_i$ where $u_i \in B$ and $v_i \in C$. Hence $w = u_1 v_1 u_2 v_2 \ldots u_n v_n$ and since each $u_i, v_i \in B + C \subseteq B^* + C^*$, it follows that $w \in (B^* + C^*)^*$.

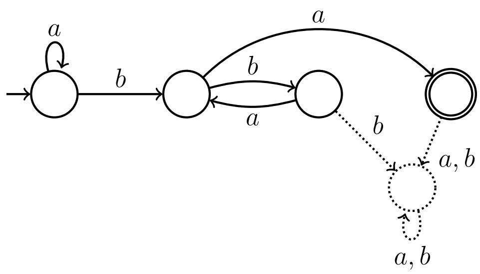
2. Let $U = (a+b), V = (a+ba)^*$ and $W = b^*$. We then have that $U \subseteq VW$ and so by the previous subpart, we have that $U^* \subseteq (V^* + W^*)^*$. Since $V^* = V$ and $W^* = W$, it follows that $(a+b)^* \subseteq ((a+ba)^* + b^*)^*$. Further, since $(a+b)^*$ is the set of all words over $\{a, b\}$, we have that $((a+ba)^* + b^*)^* \subseteq (a+b)^*$. The desired claim then follows.

Solution 1.3

1. NFA accepting L



2. DFA accepting L



Solution 1.4

1. We define $\uparrow r$ by induction on the regular expression r:

   - If $r = \emptyset$, then we set $\uparrow r = \emptyset$
   - If $r = \epsilon$, then we set $\uparrow r = \Sigma^*$
   - If $r = x$ for some $x \in \{a, b\}$, then we set $\uparrow r = \Sigma^* x \Sigma^*$
   - If $r = r_1 + r_2$ for some r_1 and r_2, then we set $\uparrow r = \ (\uparrow r_1) + (\uparrow r_2)$
   - If $r = r_1 r_2$ for some r_1 and r_2, then we set $\uparrow r = \ (\uparrow r_1)(\uparrow r_2)$
   - If $r = (r_1)^*$ for some r_1, then we set $\uparrow r = \Sigma^*$. Note that if $r = (r_1)^*$ for some r_1, then $\epsilon \in \mathcal{L}(r)$ and so $\uparrow \mathcal{L}(r)$ must contain every word.

2. Let A be an NFA recognizing a language L. We construct the NFA B from A as follows: Corresponding to every state q of A and every letter $x \in \{a, b\}$, we add a self-loop transition (q, x, q). These new transitions will be referred to as *special transitions*. We now claim that $\mathcal{L}(B) = \uparrow L$.

   Suppose $w \in \uparrow L$. Hence, $w = u_1 a_1 u_2 a_2 \ldots u_n a_n u_{n+1}$ for some words $u_1, \ldots, u_{n+1}$ and letters $a_1, \ldots, a_n$ such that $w' := a_1 a_2 \ldots a_n \in L$. Hence, there is an accepting run $\rho := q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \ldots q_{n-1} \xrightarrow{a_n} q_n$ of

A on the word w'. Now, notice that $q_0 \xrightarrow{u_1} q_0 \xrightarrow{a_1} q_1 \xrightarrow{u_2} q_1 \xrightarrow{a_2} q_1 \dots q_{n-1} \xrightarrow{a_n} q_n \xrightarrow{u_{n+1}} q_n$ is an accepting run of B on the word w. (Here $q_i \xrightarrow{u_{i+1}} q_i$ denotes that starting from the state q_i, there is a run on the word u_{i+1} which ends at q_i). This implies that $w \in \mathcal{L}(B)$.

Suppose ρ is an accepting run of B on the word w. We now prove that $w \in \uparrow L$ by induction on the number of special transitions of ρ. If ρ has no special transitions, then ρ is also a run of A on w and so $w \in L \subseteq \uparrow L$. For the induction step, suppose ρ has $k+1$ special transitions for some $k \geq 0$. Let $w = w_1 w_2 \dots w_n$ with each $w_i \in \Sigma$ and let $\rho = q_0 \xrightarrow{w_0} q_1 \xrightarrow{w_1} q_2 \dots q_{n-1} \xrightarrow{w_n} q_n$. Let $q_i \xrightarrow{w_{i+1}} q_{i+1}$ be the first special transition along ρ. Since this is a special transition, it must be the case that $q_i = q_{i+1}$. Let w' be the word obtained from w by deleting the letter w_{i+1} at the $(i+1)^{th}$ position and let ρ' be the accepting run of B on w' obtained from ρ by deleting the transition $q_i \xrightarrow{w_{i+1}} q_i$. Since ρ' has only k special transitions, by induction hypothesis, $w' \in \uparrow L$. Since w can be obtained from w' by adding a letter, it follows that $w \in \uparrow L$ as well, thereby finishing the proof.