Chair of Theoretical Computer Science and Software Reliability
Informatik
Technical University of Munich

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Registration number

Signature

Note:
- Cross your Registration number(with leading zero). It will be evaluated automatically.
- Sign in the corresponding signature field.

Automaten und formale Sprachen

Exam: IN2041 / Retake **Date:** Tuesday 4th April, 2023
Examiner: Prof. Javier Esparza **Time:** 17:00 – 19:00

	P 1	P 2	P 3	P 4	P 5	P 6	P 7	P 8
I								

Working instructions

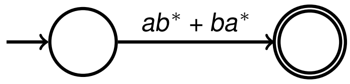
- This exam consists of **18 pages** with a total of **8 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 45 credits.
- Detaching pages from the exam is prohibited.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from __________ to __________ / Early submission at __________

Problem 1 NFAs and regular expressions (6 credits)

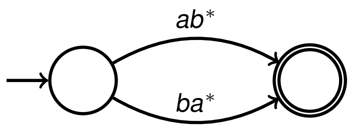
Let NFA-regtoNFA-ϵ be the algorithm given in the lectures, which given an NFA-reg M as input produces as output an NFA-ϵ which recognizes the same language as M.

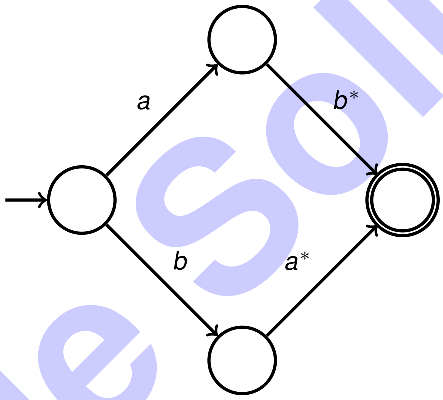
a) Let A be the following NFA-reg over the alphabet $\Sigma = \{a, b\}$.

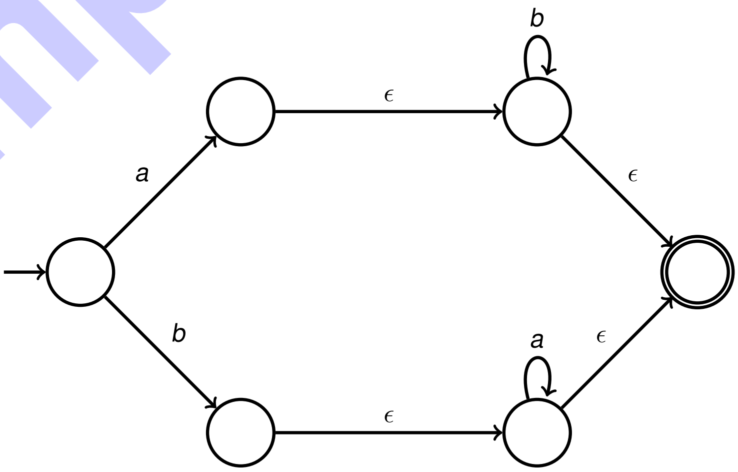


Apply the NFA-regtoNFA-ϵ algorithm on A to produce an NFA-ϵ B.

We obtain B by the following steps.







b) Consider the NFA-ϵ B from the previous subproblem. Apply any algorithm which converts an NFA-ϵ to an NFA recognizing the same language (for example, the NFA-ϵtoNFA algorithm given in the lectures), on the NFA-ϵ B, to produce an NFA C.

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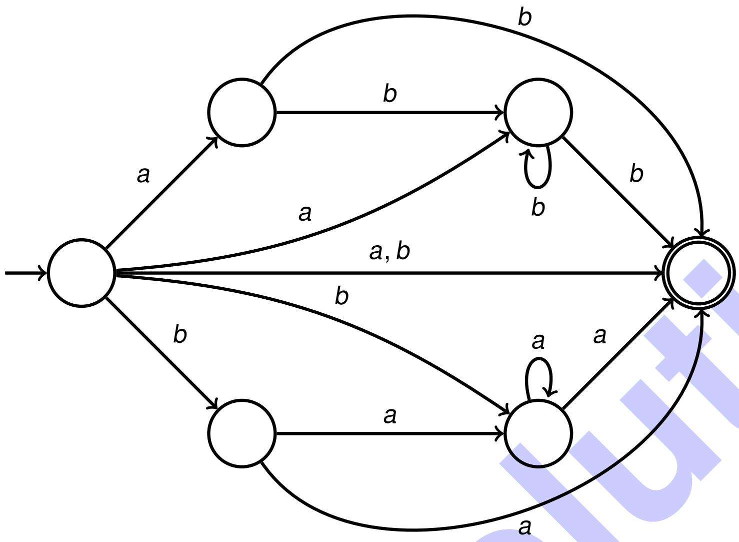
1
2

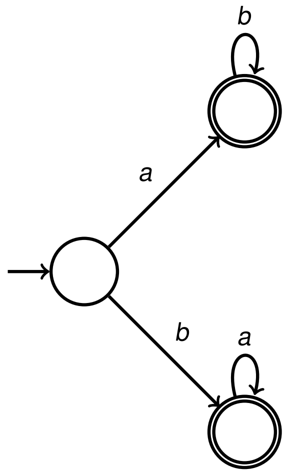
The following is one possible solution for C.



0 1 2

c) Give a minimal NFA recognizing $\mathcal{L}(ab^* + ba^*)$. Note that you have to produce such an NFA and prove that any NFA which has strictly less states cannot recognize $\mathcal{L}(ab^* + ba^*)$.

The following is a minimal NFA for $\mathcal{L}(ab^* + ba^*)$.



We now prove that there is no 2-state NFA that can recognize $L = \mathcal{L}(ab^* + ba^*)$. For the sake of contradiction, suppose D is a 2-state NFA which recognizes L. Let q be some initial state of D. q cannot be final as otherwise D accepts $\epsilon \notin L$. D must have a final state, as otherwise D accepts nothing. Let $q' \neq q$ be a final state of D. Note that q' also cannot be initial as otherwise D accepts ϵ. Hence, we have exactly one initial state q and one final state $q' \neq q$.

Since $a, b \in L$, it follows that $q \xrightarrow{a} q'$ and $q \xrightarrow{b} q'$ are transitions of D. Further since $ab \in L$, it follows that $q \xrightarrow{a} p \xrightarrow{b} q'$ for some $p \in \{q, q'\}$. If $p = q$, then aa is accepted by D because of the run $q \xrightarrow{a} q \xrightarrow{a} q'$. If $p = q'$, then bb is accepted by D because of the run $q \xrightarrow{b} q' \xrightarrow{b} q'$. In either case, we have a contradiction.

Remark: We note that a residual-based argument does not work here, as it works only for DFAs.

Problem 2 Occurrences of subwords (5 credits)

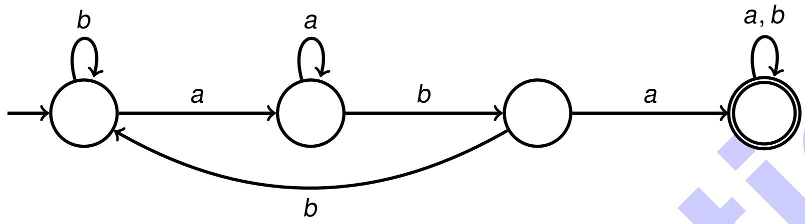
Let $\Sigma = \{a, b\}$.

a) Let L be the language of finite words over Σ defined as

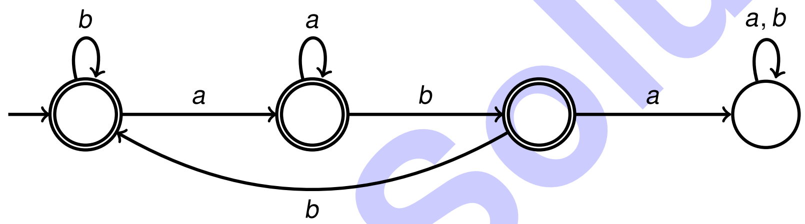
$$L = \{w : w \text{ contains } \textbf{no} \text{ occurrence of } aba\}$$

Give the minimal DFA for the language L. **Hint:** It might help to think in terms of pattern matching.

We consider the minimal DFA obtained by the pattern matching algorithm for the pattern $p = aba$, with the difference that the final state has self-loops for $\Sigma = \{a, b\}$.



Note that this is the minimal DFA for $\Sigma^* aba \Sigma^*$. Flipping the accepting and rejecting states gives the miminal DFA for L.



b) Let L' be the language of finite words over Σ defined as

$$L' = \{w : w \text{ contains } \textbf{at least two distinct} \text{ (but possibly overlapping) occurrences of } aba\}$$

For example, the words $b\,\underbrace{aba}\,a\,\underbrace{aba}\,a\,\underbrace{aba}$, $\overbrace{ab\underbrace{a}}\underbrace{ba} \in L'$ but $ab, abaab \notin L'$.

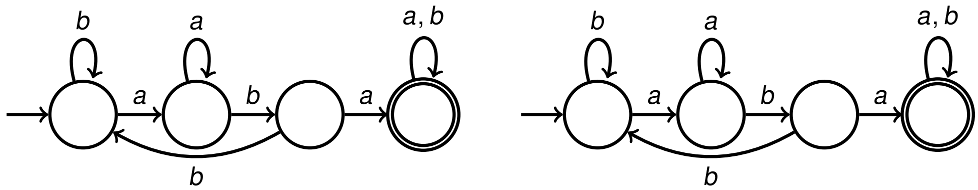
Give a regular expression for the language L'.

One possible solution is $\Sigma^* aba \Sigma^* aba \Sigma^* + \Sigma^* ababa \Sigma^*$.

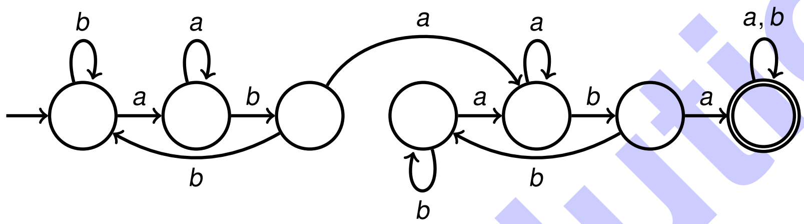
0 1 2 3

c) Give the minimal DFA for the language L' defined in the previous subproblem. **Hint:** It might help to think in terms of pattern matching. The final answer should have 7 states.

Consider two copies of the minimal DFA for the pattern $p = aba$, with the difference that the final state has self-loops for $\Sigma = \{a, b\}$.



We remove the accepting state in the first copy and divert its incoming a transition to the second state of the second copy, to get our required answer.

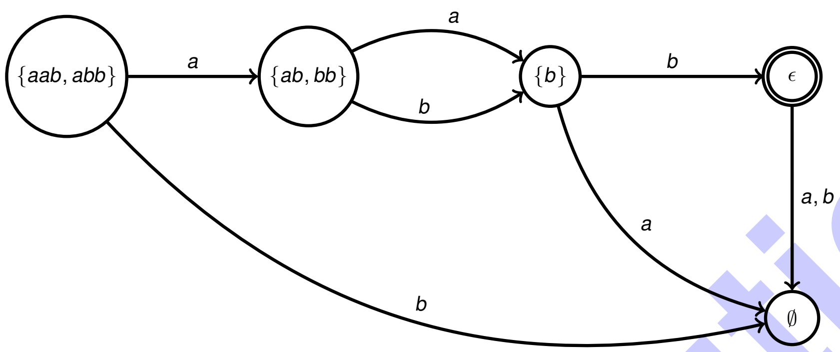


Problem 3 Fixed-length languages (4 credits)

For a fixed-length language L over $\Sigma = \{a, b\}$ we denote by q_L the state of the master automaton representing L. We also denote by $M(L)$ the fragment of the master automaton that contains q_L and all its residuals, that is, it contains all the states between q_L and $q_\emptyset$, **including q_L and $q_\emptyset$** (and no other states).

0 1 2 3 4

How many fixed-length languages L of length 3 exist such that $M(L)$ contains exactly 5 states? **For instance, here is an example of a language L of length 3 such that $M(L)$ contains exactly 5 states.**



If there are 5 states in $M(L)$, 2 of them must be in level 0 (those are $q_\emptyset$ and q_ε) and in every other layer there is exactly 1 state. A transition from the state in level i can either go to the state in level $i-1$ or to $q_\emptyset$, with the restriction that at least one transition must go to the state in level $i-1$. Hence, the edge between consecutive levels can be labeled either with a or with b or with a, b. Since we have 3 levels and 3 options for each level, there are in total $3^3 = 27$ different languages.

Problem 4 First-order logic on words (6 credits)

Let $\Sigma = \{0, 1\}$ and let $n \geq 1$ be some natural number. Given a string $w \in \Sigma^*$, let $\text{msbf}(w)$ denote the number represented by w in binary in the most significant bit first encoding. For example, if $w = 0011$, then $\text{msbf}(w) = 3$ and if $w = 1011$, then $\text{msbf}(w) = 11$.

For the purposes of this exercise, whenever you are asked to construct a formula over $\text{FO}(\Sigma)$, in addition to the syntax of $\text{FO}(\Sigma)$, you are only allowed to use the following macros: $\text{first}(x)$, $\text{last}(x)$, $x = y$, $y = x + k$, $y < x + k$ and $y < k$ for some number k. *If you use any other macros, you have to explicitly give the FO formulas that these macros stand for.*

0 1

a) For $n \geq 1$, consider the language $L_n := \{w : w \in \Sigma^{2n}\}$. Give a formula ϕ_n over $\text{FO}(\Sigma)$ which recognizes L_n. The formula ϕ_n must be of size polynomial in n, i.e., there must be a polynomial p such that the size of each ϕ_n is at most $p(n)$.

One possible solution is

$$\phi_n := \exists x, y.\ \text{first}(x) \wedge \text{last}(y) \wedge y = 2n - 1 + x$$

Intuitively, this formula states that there are two positions x and y such that x is the first position, y is the last position and the distance between them is $2n$.

0 1 2

b) For $n \geq 1$, consider the language $L'_n := \{uu : u \in \Sigma^n\}$. Give a formula ϕ'_n over $\text{FO}(\Sigma)$ which recognizes L'_n. The formula ϕ'_n must be of size polynomial in n, i.e., there must be a polynomial p' such that the size of each ϕ'_n is at most $p'(n)$.

One possible solution is

$$\phi'_n := \phi_n \wedge \forall x.\ x < n + 1 \implies \exists y.\ (y = x + n \wedge (Q_0(x) \iff Q_0(y)) \wedge (Q_1(x) \iff Q_1(y)))$$

Intuitively, this formula states that the word has length exactly $2n$ and further for every position $x < n+1$, the letter at position x is the same as the letter at position $y = x + n$.

c) For $n \geq 1$, consider the language $L''_n := \{uv : u, v \in \Sigma^n,\ \text{msbf}(u) \geq \text{msbf}(v)\}$. Give a formula ϕ''_n over FO(Σ) which recognizes L''_n. The formula ϕ''_n must be of size polynomial in n, i.e., there must be a polynomial p'' such that the size of each ϕ''_n is at most $p''(n)$.

0 1 2 3

One possible solution is

$$\phi''_n := \phi_n \wedge (\phi'_n \vee (\exists x, y.\ \ x < n+1 \wedge y = x+n \wedge Q_1(x) \wedge Q_0(y) \wedge$$
$$(\forall x'.\ x' < x \implies \exists y'.\ (y' = x'+n \wedge (Q_0(x') \iff Q_0(y')) \wedge (Q_1(x') \iff Q_1(y'))))))$$

Intuitively, this formula states that the word has length exactly $2n$ and

- Either ϕ'_n holds, in which case the word is of the form uu with $u \in \Sigma^n$
- Or there is a position $x < n+1$ such that the letter at x is 1 and the letter at $y = x+n$ is 0 and for every $x' < x$, the letter at x' and the letter at $y' = x'+n$ are the same.

Problem 5 Operations on languages (6 credits)

Let $\Sigma = \{a, b\}$. Let $L \subseteq \Sigma^*$ be any language consisting of finite words over Σ. We define the ω-language $La^\omega \subseteq \Sigma^\omega$ as

$$La^\omega = \{wa^\omega : w \in L\}$$

Note that La^ω is a language of *infinite words* over Σ. Intuitively, each word in La^ω is obtained by first taking some finite word $w \in L$ and then adding the infinite suffix a^ω to it.

0 a) Prove or disprove: If $L \subseteq \Sigma^*$ is regular, then La^ω is ω-regular.

1

The claim is true. Suppose L is a regular language. Let r be a regular expression for L. Then, the ω-regular expression $r \cdot \{a\}^\omega$ recognizes La^ω.

0 b) Prove or disprove: If La^ω is ω-regular for some $L \subseteq \Sigma^*$, then L is regular.

1

2

The claim is false. Let L be any non-regular language over $\{a\}$, for example $\{a^{2^n} : n \geq 1\}$. Then $La^\omega = a^\omega$ which is ω-regular.

c) Prove or disprove: If $(L \cdot \{b\})a^\omega$ is ω-regular for some $L \subseteq \Sigma^*$, then L is regular.

0 1 2 3

The claim is true. Suppose $L' := (L \cdot \{b\})a^\omega$ is ω-regular. Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NBA which recognizes L'. Let Q' be the set of states of A which accept ba^ω. Let B be the NFA given by $B = (Q, \Sigma, \delta, Q_0, Q')$. We claim that B recognizes L.

Suppose $w \in L$. Then there is an accepting run for wba^ω over A. Let q be the state that is reached along this run after reading w. By definition, $q \in Q'$ and so it follows that w is also accepted over B.

Suppose w is accepted by B. Then there is an accepting run of w over B which ends in some state in Q'. By definition, this means that there is an accepting run for wba^ω over A and so $wba^\omega \in L' = (L \cdot \{b\})a^\omega$. Hence, $wba^\omega = w'ba^\omega$ for some $w' \in L$. If w is a strict prefix of w', then let $w' = ww''$ for some $w'' \neq \epsilon$. We then have $ba^\omega = w''ba^\omega$, which leads to a contradiction. A similar argument can be made for the case of w' being a strict prefix of w. It follows then that $w = w'$ and so $w \in L$.

Problem 6 Acceptance conditions (10 credits)

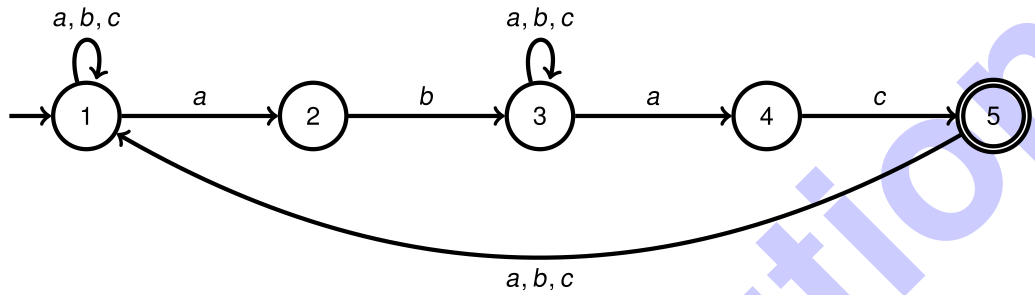
Throughout this exercise, we will only be considering languages of infinite words over $\Sigma = \{a, b, c\}$.

a) Consider the ω-regular language L_1 defined as

$$L_1 = \{w \in \Sigma^\omega : ab \text{ and } ac \text{ appear infinitely often in } w\}$$

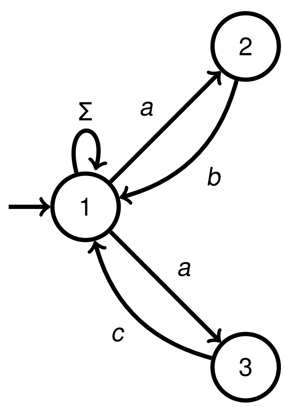
Give a non-deterministic Büchi automaton $(A_1, \mathcal{F}_1)$ which accepts L_1 such that A_1 has **at most 5 states**.

The following is one possible solution.



b) Give a non-deterministic generalized Büchi automaton $(A_1', \mathcal{F}_1')$ which accepts L_1 such that A_1' has **at most 3 states**.

The following is one possible solution whose generalized Büchi condition is $\{\{2\}, \{3\}\}$.



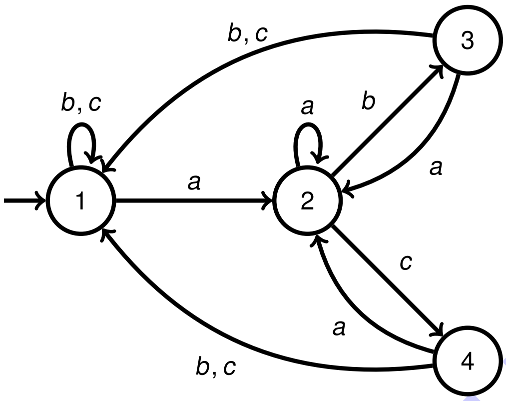
c) Consider the ω-regular language L_2 defined as

$$L_2 = \{w \in \Sigma^\omega : ab \text{ appears infinitely often in } w \text{ and } ac \text{ appears } \textbf{finitely} \text{ often in } w\}$$

Give a **deterministic** Rabin automaton $(A_2, \mathcal{F}_2)$ which accepts L_2 such that A_2 has **at most 4 states**.

0 1 2 3

The following is one possible solution whose Rabin condition is $\{\langle\{3\},\{4\}\rangle\}$, i.e., 3 must be visited infinitely often and 4 must be visited finitely often.

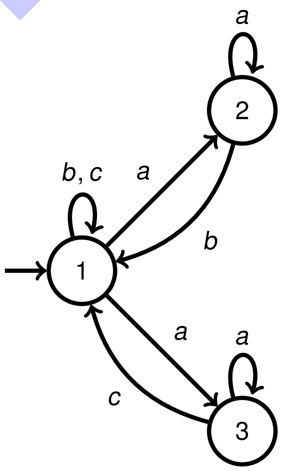


d) **This is a bonus subproblem.**

Give a non-deterministic Muller automaton $(A_2', \mathcal{F}_2')$ which accepts L_2 such that A_2' has **at most 3 states**.

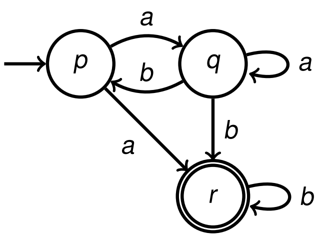
0 1 2 3

The following is one possible solution whose Muller condition is $\{\{1, 2\}\}$.



Problem 7 DAGs and Büchi automata (4 credits)

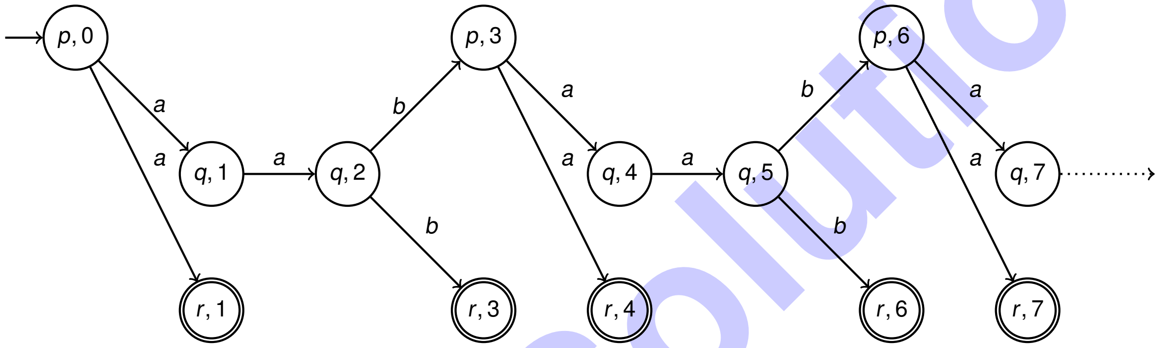
Consider the following Büchi automaton over $\Sigma = \{a, b\}$.



0 1 2

a) Draw $dag((aab)^\omega)$ and give an odd ranking for it.

The $dag((aab)^\omega)$ is presented below.



One possible way to define an odd ranking is

$$f(s, i) = \begin{cases} 1 & \text{if } (s = p \text{ or } s = q) \text{ and } \langle s, i\rangle \text{ appears in dag}((aab)^\omega), \\ 0 & \text{if } s = r \text{ and } \langle s, i\rangle \text{ appears in dag}((aab)^\omega), \\ \perp & \text{otherwise.} \end{cases}$$

b) Find an ω-word w such that $dag(w)$ **does not** have an odd ranking. Draw $dag(w)$ and prove that it does not have an odd ranking by analyzing the *dag*.

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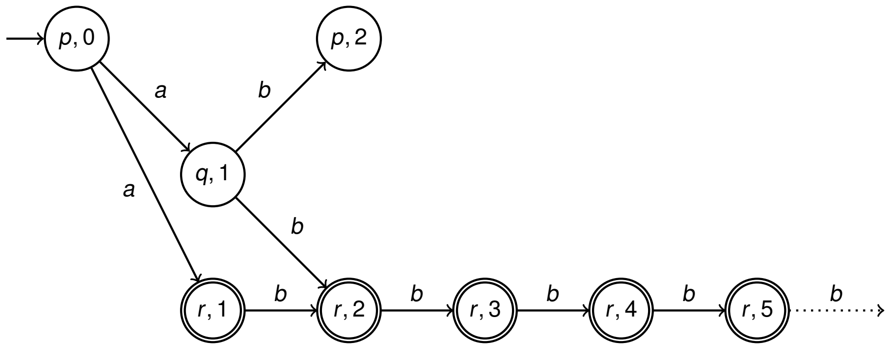
For example, ab^ω has this property. Below we sketch $dag(ab^\omega)$.



There are only two infinite paths in this dag and starting from layer 2 both of them visit only the state r which is accepting. Hence, there can be no odd ranking in this case.

Problem 8 Linear Temporal Logic (4 credits)

0 1 2

a) Let $AP = \{p, q, r\}$ and let $\Sigma = 2^{AP}$. Consider the formulas

$$\phi := (p \mathbf{U} q) \mathbf{U} r \quad \text{and} \quad \xi := p \mathbf{U} (q \mathbf{U} r)$$

Give four computations $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, all of them over AP, such that

- $\sigma_1 \models \phi$ and $\sigma_1 \models \xi$
- $\sigma_2 \models \phi$ and $\sigma_2 \not\models \xi$
- $\sigma_3 \not\models \phi$ and $\sigma_3 \models \xi$
- $\sigma_4 \not\models \phi$ and $\sigma_4 \not\models \xi$

There are many possible solutions, here is an example:

- $\sigma_1 = \{r\}^\omega$
- $\sigma_2 = \{p\}\{q\}\{p\}\{q\}\{r\}^\omega$
- $\sigma_3 = \{p\}\{r\}^\omega$
- $\sigma_4 = \emptyset^\omega$

0 1

b) **This is a bonus subproblem.**

Let $AP = \{p, q\}$ and let $\Sigma = 2^{AP}$. Give an ω-regular expression over Σ for the set of all computations which satisfy the formula

$$\varphi := (p \mathbf{U} q) \mathbf{U} p$$

$$\{q\}^*(\{p\} + \{p, q\})\Sigma^\omega$$

c) **This is a bonus subproblem.**

0
1

Consider the formula φ defined in the previous subproblem. Use the ω-regular expression you defined in the previous subproblem to derive a formula φ' such that φ' and φ are equivalent and φ' is of strictly smaller size than φ.

$q \; \mathbf{U} \; p$

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.