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Automata and Formal Languages

Winter Term 2023/24 – Exercise Sheet 13

Exercise 13.1.

Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. Give LTL formulas for the following ω-languages:

(a) $\{p, q\}\, \emptyset\, \Sigma^\omega$

(b) $\Sigma^*\, \{q\}^\omega$

(c) $\Sigma^*\, (\{p\} + \{p, q\})\, \Sigma^*\, \{q\}\, \Sigma^\omega$

(d) $\{p\}^*\, \{q\}^*\, \emptyset^\omega$

In (a) and (d) the $\emptyset$ symbol stands for the letter $\emptyset \in \Sigma$, and not for the empty ω-language.

Solution.

(a) $(p \wedge q) \wedge \mathbf{X}(\neg p \wedge \neg q)$

(b) $\mathbf{FG}(\neg p \wedge q)$

(c) $\mathbf{F}(p \wedge \mathbf{XF}(\neg p \wedge q))$

(d) $(p \wedge \neg q)\ \mathbf{U}\ ((\neg p \wedge q)\ \mathbf{U}\ \mathbf{G}(\neg p \wedge \neg q))$

Exercise 13.2.

Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. Give Büchi automata for the ω-languages over Σ defined by the following LTL formulas:

(a) $\mathbf{XG}\neg p$

(b) $(\mathbf{GF}p) \to (\mathbf{F}q)$

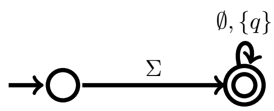
(c) $p \wedge \neg(\mathbf{XF}p)$

(d) $\mathbf{G}(p\ \mathbf{U}\ (p \to q))$

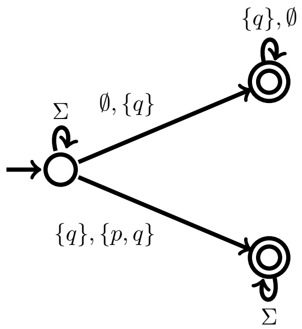
(e) $\mathbf{F}q \to (\neg q\ \mathbf{U}\ (\neg q \wedge p))$

Solution.

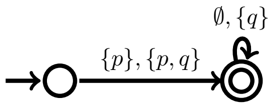
(a)



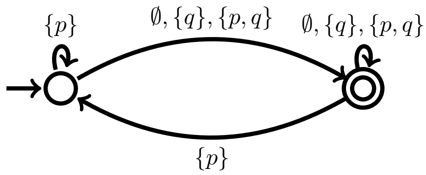
(b) Note that $(\mathbf{GF}p) \to (\mathbf{F}q) \equiv \neg(\mathbf{GF}p) \vee (\mathbf{F}q) \equiv (\mathbf{FG}\neg p) \vee (\mathbf{F}q)$. We construct Büchi automata for $\mathbf{FG}\neg p$ and $\mathbf{F}q$, and take their union:



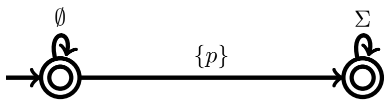
(c) Note that $p \wedge \neg(\mathbf{XF}p) \equiv p \wedge \mathbf{XG}\neg p$. We construct a Büchi automaton for $p \wedge \mathbf{XG}\neg p$:



(d)



(e) Note that $\mathbf{F}q \rightarrow (\neg q \; \mathbf{U} \; (\neg q \wedge p)) \equiv \mathbf{G}\neg q \vee (\neg q \; \mathbf{U} \; (\neg q \wedge p))$. Consider this case split over the occurence of a p : computations that satisfy the formula either have no occurrence of p, in which case they must satisfy the first part of the $\vee$ (i.e. $\mathbf{G}\neg q$), or they have a first occurrence of p with no q before or at the same time:



Exercise 13.3.

Say which of the following equivalences hold. For every equivalence that does not hold give an instantiation of φ and ψ together with a computation that disproves the equivalence.

(a) $\mathbf{F}(\varphi \vee \psi) \equiv \mathbf{F}\varphi \vee \mathbf{F}\psi$

(b) $\mathbf{F}(\varphi \wedge \psi) \equiv \mathbf{F}\varphi \wedge \mathbf{F}\psi$

(c) $\mathbf{G}(\varphi \vee \psi) \equiv \mathbf{G}\varphi \vee \mathbf{G}\psi$

(d) $(\varphi \vee \psi) \; \mathbf{U} \; \rho \equiv (\varphi \; \mathbf{U} \; \rho) \vee (\psi \; \mathbf{U} \; \rho)$

(e) $\mathbf{GF}(\varphi \wedge \psi) \equiv \mathbf{GF}\varphi \wedge \mathbf{GF}\psi$

(f) $\mathbf{X}(\varphi \; \mathbf{U} \; \psi) \equiv (\mathbf{X}\varphi \; \mathbf{U} \; \mathbf{X}\psi)$

Solution.

(a) True, since:

$$\begin{aligned} \sigma \models \mathbf{F}(\varphi \vee \psi) &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \vee \psi) \\ &\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi) \vee (\sigma^k \models \psi) \\ &\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi) \vee (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi) \\ &\iff \sigma \models \mathbf{F}\varphi \vee \mathbf{F}\psi. \end{aligned}$$

(b) False. Let $\sigma = \{p\}\{q\}\emptyset^\omega$. We have $\sigma \models \mathbf{F}p \wedge \mathbf{F}q$ and $\sigma \not\models \mathbf{F}(\varphi \wedge \psi)$.

(c) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \models \mathbf{G}(p \vee q)$ and $\sigma \not\models \mathbf{G}p \vee \mathbf{G}q$.

(d) False. Let $\sigma = \{p\}\{q\}\{r\}\emptyset^\omega$. We have $\sigma \models (p \vee q)\ \mathbf{U}\ r$ and $\sigma \not\models (p\ \mathbf{U}\ r) \vee (q\ \mathbf{U}\ r)$.

(e) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \not\models \mathbf{GF}(p \wedge q)$ and $\sigma \models \mathbf{GF}p \wedge \mathbf{GF}q$.

(f) True, since:

$$\begin{aligned}
\sigma \models \mathbf{X}(\varphi\ \mathbf{U}\ \psi) &\iff \sigma^1 \models (\varphi\ \mathbf{U}\ \psi) \\
&\iff \exists k \geq 0 : (\sigma^1)^k \models \varphi \text{ and } \forall 0 \leq i < k\ (\sigma^1)^i \models \psi \\
&\iff \exists k \geq 0 : (\sigma^k)^1 \models \varphi \text{ and } \forall 0 \leq i < k\ (\sigma^i)^1 \models \psi \\
&\iff \exists k \geq 0 : \sigma^k \models \mathbf{X}\varphi \text{ and } \forall 0 \leq i < k\ (\sigma^i \models \mathbf{X}\psi) \\
&\iff \sigma \models (\mathbf{X}\varphi)\ \mathbf{U}\ (\mathbf{X}\psi).
\end{aligned}$$

Exercise 13.4.

Let $\text{AP} = \{p, q\}$ and let $\Sigma = 2^{\text{AP}}$. An LTL formula is a tautology if it is satisfied by all computations. Which of the following LTL formulas are tautologies?

(a) $\mathbf{G}p \to \mathbf{F}p$

(b) $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$

(c) $\mathbf{FG}p \vee \mathbf{FG}\neg p$

(d) $\neg\mathbf{F}p \to \mathbf{F}\neg\mathbf{F}p$

(e) $(\mathbf{G}p \to \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \vee q))$

(f) $\neg(p\ \mathbf{U}\ q) \leftrightarrow (\neg p\ \mathbf{U}\ \neg q)$

(g) $\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p)$

Solution.

(a) $\mathbf{G}p \to \mathbf{F}p$ is a tautology since

$$\begin{aligned}
\sigma \models \mathbf{G}p &\iff \forall k \geq 0\ \sigma^k \models p \\
&\implies \exists k \geq 0\ \sigma^k \models p \\
&\iff \sigma \models \mathbf{F}p.
\end{aligned}$$

(b) $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ is a tautology. For the sake of contradiction, suppose this is not the case. There exists σ such that

$$\sigma \models \mathbf{G}(p \to q), \text{ and} \quad (1)$$
$$\sigma \not\models (\mathbf{G}p \to \mathbf{G}q). \quad (2)$$

By (2), we have

$$\begin{aligned}
&\sigma \models \mathbf{G}p, \text{ and} \\
&\sigma \not\models \mathbf{G}q.
\end{aligned}$$

Therefore, there exists $k \geq 0$ such that $p \in \sigma(k)$ and $q \notin \sigma(k)$ which contradicts (1).

(c) $\mathbf{FG}p \vee \mathbf{FG}\neg p$ is not a tautology since it is not satisfied by $(\{p\}\{q\})^\omega$.

(d) $\neg\mathbf{F}p \to \mathbf{F}\neg\mathbf{F}p$ is a tautology since $\varphi \to \mathbf{F}\varphi$ is a tautology for every formula φ.

(e) $(\mathbf{G}p \to \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \vee q))$ is a tautology. We have

$$\begin{aligned}
\mathbf{G}p \to \mathbf{F}q &\equiv \neg\mathbf{G}p \vee \mathbf{F}q && \text{(by def. of implication)} \\
&\equiv \mathbf{F}\neg p \vee \mathbf{F}q \\
&\equiv \mathbf{F}(\neg p \vee q) \\
&\equiv \mathbf{F}(p \to q) && \text{(by def. of implication)}
\end{aligned}$$

Therefore, we have to show that

$$\mathbf{F}(p \to q) \leftrightarrow (p\ \mathbf{U}\ (p \to q)).$$

$\leftarrow$) Let σ be such that $\sigma \models (p\ \mathbf{U}\ (p \to q))$. In particular, there exists $k \geq 0$ such that $\sigma^k \models (p \to q)$. Therefore, $\sigma \models \mathbf{F}(p \to q)$.

$\to$) Let σ be such that $\sigma \models \mathbf{F}(p \to q)$. Let $k \geq 0$ be the smallest position such that $\sigma^k \models (p \to q)$. For every $0 \leq i < k$, we have $\sigma^i \not\models (p \to q)$ which is equivalent to $\sigma^i \models p \wedge \neg q$. Therefore, for every $0 \leq i < k$, we have $\sigma^i \models p$. This implies that $\sigma \models p\ \mathbf{U}\ (p \to q)$.

(f) $\neg(p\ \mathbf{U}\ q) \leftrightarrow (\neg p\ \mathbf{U}\ \neg q)$ is not a tautology. Let $\sigma = \{p\}\{q\}^\omega$. We have $\sigma \not\models \neg(p\ \mathbf{U}\ q)$ and $\sigma \models (\neg p\ \mathbf{U}\ \neg q)$.

(g) $\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p)$ is a tautology since

$$\begin{aligned}
\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p) &\equiv \neg\mathbf{G}(\neg p \vee \mathbf{X}p) \vee (\neg p \vee \mathbf{G}p) && \text{(by def. of implication)} \\
&\equiv \mathbf{F}(p \wedge \neg\mathbf{X}p) \vee \neg p \vee \mathbf{G}p \\
&\equiv \neg\mathbf{G}p \to (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) && \text{(by def. of implication)} \\
&\equiv \mathbf{F}\neg p \to (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) \\
&\equiv \mathbf{F}\neg p \to \mathbf{F}\neg p.
\end{aligned}$$