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Automata and Formal Languages

Winter Term 2023/24 – Exercise Sheet 9

Exercise 9.1.

Let $\Sigma := \{a, b\}$. Construct the following MSO(Σ) formulae.

(a) $\mathrm{Pos}_a(X)$: X contains exactly the positions with an a.

(b) $\mathrm{Between}(X, Y)$: between every two elements of X there is an element of Y.

(c) $\mathrm{Without}(X, Y, Z)$: $Z = X \setminus Y$.

(d) $\mathrm{Min}(X, x)$, $\mathrm{Max}(X, x)$: x is the minimum/maximum of X.

(e) $\mathrm{EvenSize}(X)$: $|X|$ is even.

(f) φ with $L(\varphi) = \{w \in \Sigma^* : |w|_a, |w|_b \text{ even}\}$.

Exercise 9.2.

Consider the logic PureMSO(Σ) with syntax

$$\varphi := X \subseteq Q_a \mid X < Y \mid X \subseteq Y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists X.\ \varphi$$

Notice that formulas of PureMSO(Σ) do not contain first-order variables. The satisfaction relation of PureMSO(Σ) is given by:

$$\begin{array}{lcll} (w, \mathcal{J}) & \models & X \subseteq Q_a & \text{iff} \quad w[p] = a \text{ for every } p \in \mathcal{J}(X) \\ (w, \mathcal{J}) & \models & X < Y & \text{iff} \quad p < p' \text{ for every } p \in \mathcal{J}(X),\ p' \in \mathcal{J}(Y) \\ (w, \mathcal{J}) & \models & X \subseteq Y & \text{iff} \quad p \in \mathcal{J}(Y) \text{ for every } p \in \mathcal{J}(X) \end{array}$$

with the rest as for MSO(Σ).

Prove that MSO(Σ) and PureMSO(Σ) have the same expressive power for sentences. That is, show that for every sentence ϕ of MSO(Σ) there is an equivalent sentence ψ of PureMSO(Σ), and vice versa.

Exercise 9.3.

Let $r \geq 0$ and $n \geq 1$. Give a Presburger formula φ such that

$$(x, y) \in \mathrm{Sol}(\varphi) \quad \text{iff} \quad x > y \text{ and } x - y \equiv r \pmod{n}$$

Give an automaton that accepts the solutions of φ for $r = 1$ and $n = 2$. (It is not necessary to use the algorithm from the lecture to construct this automaton.)