Exercise 9.1.

Let $\Sigma := \{a, b\}$. Construct the following MSO($\Sigma$) formulae.

(a) $\text{Pos}_a(X)$: $X$ contains exactly the positions with an $a$.
(b) $\text{Between}(X, Y)$: between every two elements of $X$ there is an element of $Y$.
(c) $\text{Without}(X, Y, Z)$: $Z = X \setminus Y$.
(d) $\text{Min}(X, x), \text{Max}(X, x)$: $x$ is the minimum/maximum of $X$.
(e) $\text{EvenSize}(X)$: $|X|$ is even.
(f) $\varphi$ with $L(\varphi) = \{w \in \Sigma^* : |w|_a, |w|_b \text{ even}\}$.

Exercise 9.2.

Consider the logic PureMSO($\Sigma$) with syntax

$$\varphi := X \subseteq Q_a \mid X < Y \mid X \subseteq Y \mid \neg \varphi \mid \varphi \lor \varphi \mid \exists X. \varphi$$

Notice that formulas of PureMSO($\Sigma$) do not contain first-order variables. The satisfaction relation of PureMSO($\Sigma$) is given by:

- $(w, J) \models X \subseteq Q_a$ iff $w[p] = a$ for every $p \in J(X)$
- $(w, J) \models X < Y$ iff $p < p'$ for every $p \in J(X), p' \in J(Y)$
- $(w, J) \models X \subseteq Y$ iff $p \in J(Y)$ for every $p \in J(X)$

with the rest as for MSO($\Sigma$).

Prove that MSO($\Sigma$) and PureMSO($\Sigma$) have the same expressive power for sentences. That is, show that for every sentence $\varphi$ of MSO($\Sigma$) there is an equivalent sentence $\psi$ of PureMSO($\Sigma$), and vice versa.

Exercise 9.3.

Let $r \geq 0$ and $n \geq 1$. Give a Presburger formula $\varphi$ such that

$$(x, y) \in \text{Sol}(\varphi) \text{ iff } x > y \text{ and } x - y \equiv r \pmod{n}$$

Give an automaton that accepts the solutions of $\varphi$ for $r = 1$ and $n = 2$. (It is not necessary to use the algorithm from the lecture to construct this automaton.)