Prof. Javier Esparza
Philipp Czerner

Technical University of Munich
Chair for Foundations of Software Reliability

Automata and Formal Languages

Winter Term 2023/24 – Exercise Sheet 7

Exercise 7.1.

Let $\mathrm{val} : \{0,1\}^* \to \mathbb{N}$ be the function that associates to every word $w \in \{0,1\}^*$ the number $\mathrm{val}(w)$ represented by w in the *least significant bit first* encoding.

(a) Give a transducer that doubles numbers, i.e. a transducer accepting

$$L_1 = \{[x, y] \in (\{0,1\} \times \{0,1\})^* \mid \mathrm{val}(y) = 2 \cdot \mathrm{val}(x)\}.$$

(b) Give an algorithm that takes $k \in \mathbb{N}$ as input, and that produces a transducer A_k accepting

$$L_k = \left\{[x, y] \in (\{0,1\} \times \{0,1\})^* \mid \mathrm{val}(y) = 2^k \cdot \mathrm{val}(x)\right\}.$$

Hint: use (a) and consider operations seen in class.

(c) Give a transducer for the addition of two numbers, i.e. a transducer accepting

$$\{[x, y, z] \in (\{0,1\} \times \{0,1\} \times \{0,1\})^* \mid \mathrm{val}(z) = \mathrm{val}(x) + \mathrm{val}(y)\}.$$

(d) For every $k \in \mathbb{N}_{>0}$, let

$$X_k = \{[x, y] \in (\{0,1\} \times \{0,1\})^* \mid \mathrm{val}(y) = k \cdot \mathrm{val}(x)\}.$$

Sketch an algorithm that takes as input transducers A and B, accepting respectively X_a and X_b for some $a, b \in \mathbb{N}_{>0}$, and that produces a transducer C accepting X_{a+b}.

(e) Let $k \in \mathbb{N}_{>0}$. Using (b) and (d), how can you build a transducer accepting X_k?

(f) Show that the following language has infinitely many residuals, and hence that it is not regular:

$$\left\{[x, y] \in (\{0,1\} \times \{0,1\})^* \mid \mathrm{val}(y) = \mathrm{val}(x)^2\right\}.$$

Exercise 7.2.

Let $L_1 = \{bba, aba, bbb\}$ and $L_2 = \{aba, abb\}$.

(a) Give an algorithm for the following operation:

INPUT:	A fixed-length language $L \subseteq \Sigma^k$ described explicitly as a set of words.
OUTPUT:	State q of the master automaton over Σ such that $L(q) = L$.

(b) Use the previous algorithm to build the states of the master automaton for L_1 and L_2.

(c) Compute the state of the master automaton representing $L_1 \cup L_2$.

(d) Identify the kernels $\langle L_1 \rangle$, $\langle L_2 \rangle$, and $\langle L_1 \cup L_2 \rangle$.

Exercise 7.3.

We define the *language* of a Boolean formula φ over variables $x_1, \dots, x_n$ as:

$$\mathcal{L}(\varphi) = \{a_1 a_2 \cdots a_n \in \{0,1\}^n : \text{ the assignment } x_1 \mapsto a_1, \dots, x_n \mapsto a_n \text{ satisfies } \varphi\}.$$

(a) Give a polynomial-time algorithm that takes as input a DFA A recognizing a language of length n, and returns a Boolean formula φ such that $\mathcal{L}(\varphi) = \mathcal{L}(A)$.

(b) Give an exponential-time algorithm that takes a Boolean formula φ as input, and returns a DFA A recognizing $\mathcal{L}(\varphi)$.

Puzzle exercise 7.4.

Let $n \in \mathbb{N}$. Construct an NFA $N = (Q, \Sigma, \delta, Q_0, F)$ with $\mathcal{O}(n)$ states, s.t. the shortest word in $\Sigma^* \setminus L(N)$ has length at least 2^n.

Notes: The puzzle exercises cover advanced material and are not directly relevant to the exam. No model solutions will be provided. You can get feedback on your solution by sending it to me (Zulip or mail), or coming to my office in person (MI 03.11.037).