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Automata and Formal Languages

Winter Term 2023/24 – Exercise Sheet 7

Exercise 7.1.

Let $\mathrm{val} : \{0,1\}^* \to \mathbb{N}$ be the function that associates to every word $w \in \{0,1\}^*$ the number $\mathrm{val}(w)$ represented by w in the *least significant bit first* encoding.

(a) Give a transducer that doubles numbers, i.e. a transducer accepting

$$L_1 = \{[x, y] \in (\{0,1\} \times \{0,1\})^* \mid \mathrm{val}(y) = 2 \cdot \mathrm{val}(x)\}\,.$$

(b) Give an algorithm that takes $k \in \mathbb{N}$ as input, and that produces a transducer A_k accepting

$$L_k = \left\{[x, y] \in (\{0,1\} \times \{0,1\})^* \mid \mathrm{val}(y) = 2^k \cdot \mathrm{val}(x)\right\}.$$

Hint: use (a) and consider operations seen in class.

(c) Give a transducer for the addition of two numbers, i.e. a transducer accepting

$$\{[x, y, z] \in (\{0,1\} \times \{0,1\} \times \{0,1\})^* \mid \mathrm{val}(z) = \mathrm{val}(x) + \mathrm{val}(y)\}\,.$$

(d) For every $k \in \mathbb{N}_{>0}$, let

$$X_k = \{[x, y] \in (\{0,1\} \times \{0,1\})^* \mid \mathrm{val}(y) = k \cdot \mathrm{val}(x)\}\,.$$

Sketch an algorithm that takes as input transducers A and B, accepting respectively X_a and X_b for some $a, b \in \mathbb{N}_{>0}$, and that produces a transducer C accepting X_{a+b}.

(e) Let $k \in \mathbb{N}_{>0}$. Using (b) and (d), how can you build a transducer accepting X_k?

(f) Show that the following language has infinitely many residuals, and hence that it is not regular:

$$\left\{[x, y] \in (\{0,1\} \times \{0,1\})^* \mid \mathrm{val}(y) = \mathrm{val}(x)^2\right\}.$$

Solution.

(a) Let $[x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_n] \in (\{0,1\} \times \{0,1\})^n$ where $n \geq 2$. Multiplying a binary number by two shifts its bits and adds a zero. For example, the word

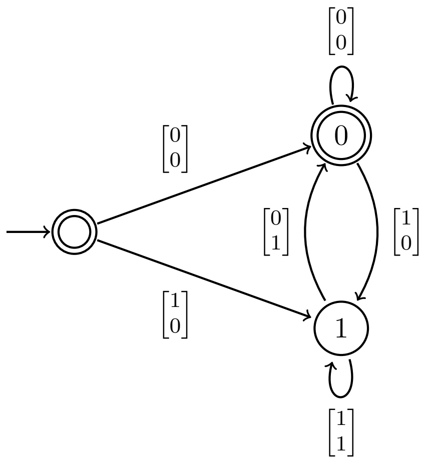
$$\begin{bmatrix} 10110 \\ 01011 \end{bmatrix}$$

belongs to the language since it encodes $[13, 26]$. Thus, we have $\mathrm{val}(y) = 2 \cdot \mathrm{val}(x)$ if and only if $y_1 = 0$, $x_n = 0$, and $y_i = x_{i-1}$ for every $1 < i \leq n$. From this observation, we construct a transducer that

- tests whether the first bit of y is 0,

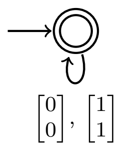
- tests whether y is consistent with x, by keeping the last bit of x in memory,
- accepts $[x, y]$ if the last bit of x is 0.

Note that words $[\varepsilon, \varepsilon]$ and $[0, 0]$ both encode the numerical values $[0, 0]$. Therefore, they should also be accepted since $2 \cdot 0 = 0$. We obtain the following transducer:



[hard] The initial state can be merged with state 0 as they have the same outgoing transitions.

(b) We construct A_0 as the following transducer accepting $\{[x, y] \in (\{0, 1\} \times \{0, 1\})^* : y = x\}$:



Let A_1 be the transducer obtained in (a). For every $k > 1$, we define $A_k = \mathit{Join}(A_{k-1}, A_1)$. A simple inductions show that $L(A_k) = L_k$ for every $k \in \mathbb{N}$.

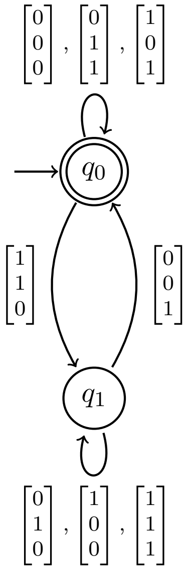
(c) We construct a transducer that computes the addition by keeping the current carry bit. Consider some tuple $[x, y, z] \in \{0, 1\}^3$ and a carry bit r. Adding x, y and r leads to the bit

$$z = (x + y + r) \bmod 2. \tag{1}$$

Moreover, it yields a new carry bit r' such that $r' = 1$ if $x + y + r > 1$ and $r' = 0$ otherwise. The following table identifies the new carry bit r' of the tuples that satisfy (1):

	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$
$r = 0$	0	x	x	0	x	0	1	x
$r = 1$	x	0	1	x	1	x	x	1

We construct our transducer from the above table:



(d) We construct a transducer C that, intuitively, feeds its input to both A and B, and then feed the respective outputs of A and B to a transducer performing addition. More formally, let $A = (Q_A, \{0,1\}, \delta_A, q_{0A}, F_A)$, $B = (Q_B, \{0,1\}, \delta_B, q_{0B}, F_B)$, and let $D = (Q_D, \{0,1\}, \delta_D, q_{0D}, F_D)$ be the transducer for addition obtained in (c). We define C as $C = (Q_C, \{0,1\}, \delta_C, q_{0C}, F_C)$ where

- $Q_C = Q_A \times Q_B \times Q_D$,
- $q_{0C} = (q_{0A}, q_{0B}, q_{0D})$,
- $F_C = F_A \times F_B \times F_D$,

and

$$\delta_C((p, p', p''), [x, z]) = \{(q, q', q'') : \exists y, y' \in \{0,1\} \\ \text{s.t. } p \xrightarrow{[x,y]}_A q, p' \xrightarrow{[x,y']}_B q' \text{ and } p'' \xrightarrow{[y,y',z]}_D q''\}.$$

(e) Let $\ell = \lceil \log_2(k) \rceil$. There exist $c_0, c_1, ..., c_\ell \in \{0,1\}$ such that $k = c_0 \cdot 2^0 + c_1 \cdot 2^1 + ... + c_\ell \cdot 2^\ell$. Let $I = \{0 \leq i \leq \ell : c_i = 1\}$. Note that $k = \sum_{i \in I} 2^i$. Therefore, we may use transducer A_i from (b) for each $i \in I$, and combine these transducers using (d).

(f) For every $n \in \mathbb{N}_{>0}$, let

$$u_n = \begin{bmatrix} 0^n 1 \\ 0^n 0 \end{bmatrix} \text{ and } v_n = \begin{bmatrix} 0^{n-1} 0 \\ 0^{n-1} 1 \end{bmatrix}.$$

Let $i, j \in \mathbb{N}_{>0}$ be such that $i \neq j$. We claim that $L^{u_i} \neq L^{u_j}$. We have

$$u_i v_i = \begin{bmatrix} 0^i 1 0^i \\ 0^{2i} 1 \end{bmatrix} \text{ and } u_j v_i = \begin{bmatrix} 0^j 1 0^i \\ 0^{i+j} 1 \end{bmatrix}.$$

Therefore, $u_i v_i$ encodes $[2^i, 2^{2i}]$, and $u_i v_j$ encodes $[2^j, 2^{i+j}]$. We observe that $u_i v_i$ belongs to the language since $2^{2i} = (2^i)^2$. However, $u_j v_i$ does not belong to the language since $2^{i+j} \neq 2^{2j} = (2^j)^2$. □

Exercise 7.2.

Let $L_1 = \{bba, aba, bbb\}$ and $L_2 = \{aba, abb\}$.

(a) Give an algorithm for the following operation:

INPUT: A fixed-length language $L \subseteq \Sigma^k$ described explicitly as a set of words.
OUTPUT: State q of the master automaton over Σ such that $L(q) = L$.

(b) Use the previous algorithm to build the states of the master automaton for L_1 and L_2.

(c) Compute the state of the master automaton representing $L_1 \cup L_2$.

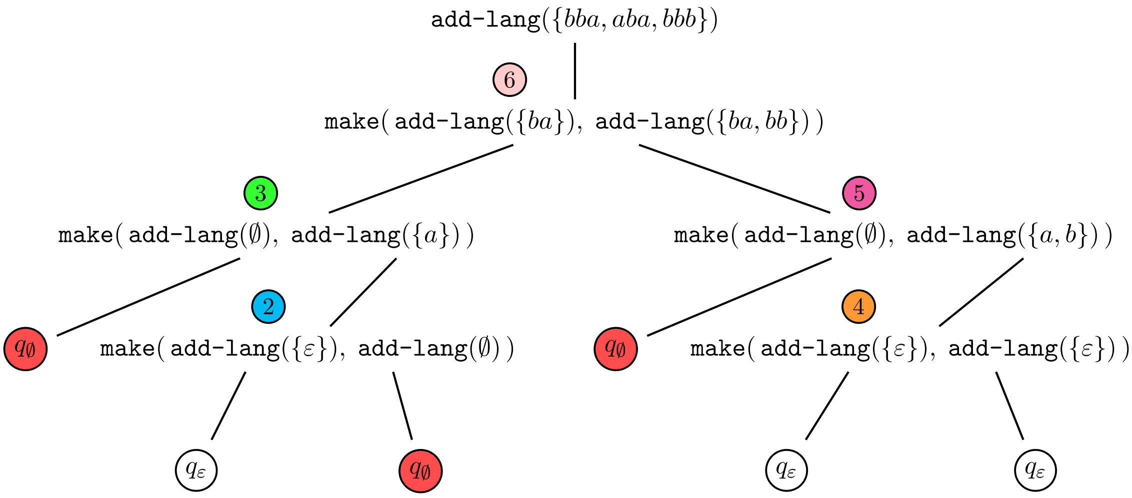
(d) Identify the kernels $\langle L_1 \rangle$, $\langle L_2 \rangle$, and $\langle L_1 \cup L_2 \rangle$.

Solution. (a)

Input: A fixed-length language $L \subseteq \Sigma^k$ described explicitely by a set of words.
Output: State q of the master automaton over Σ such that $L(q) = L$.

`add-lang`(L):
    if $L = \emptyset$ **then**
        return $q_\emptyset$
    else if $L = \{\varepsilon\}$ **then**
        return q_ε
    else
        for $a_i \in \Sigma$ **do**
            $L^{a_i} \leftarrow \{u \mid a_i u \in L\}$
            $s_i \leftarrow$ `add-lang`(L^{a_i})
        return `make`$(s_1, s_2, \ldots, s_n)$

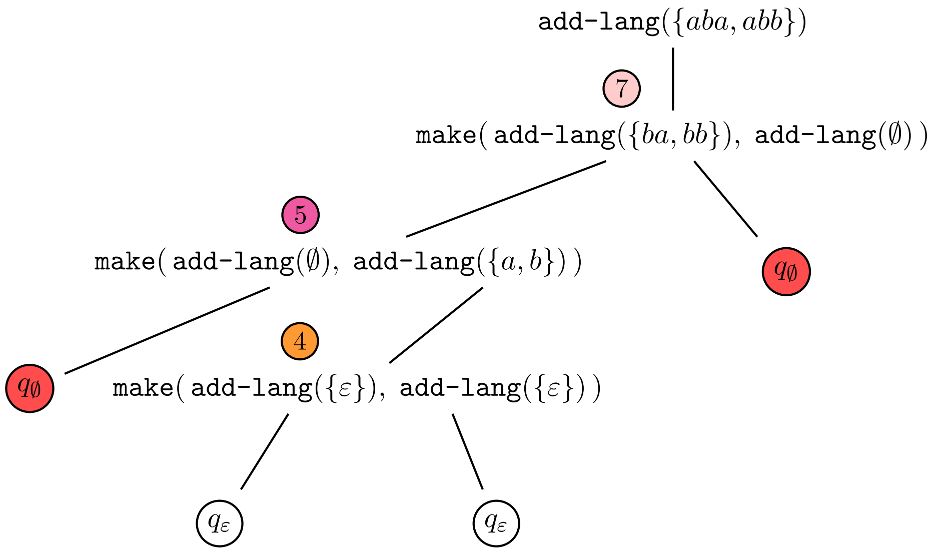
(b) Executing `add-lang`(L_1) yields the following computation tree:



The table obtained after the execution is as follows:

Ident.	a-succ	b-succ
2	q_ε	$q_\emptyset$
3	$q_\emptyset$	2
4	q_ε	q_ε
5	$q_\emptyset$	4
6	3	5

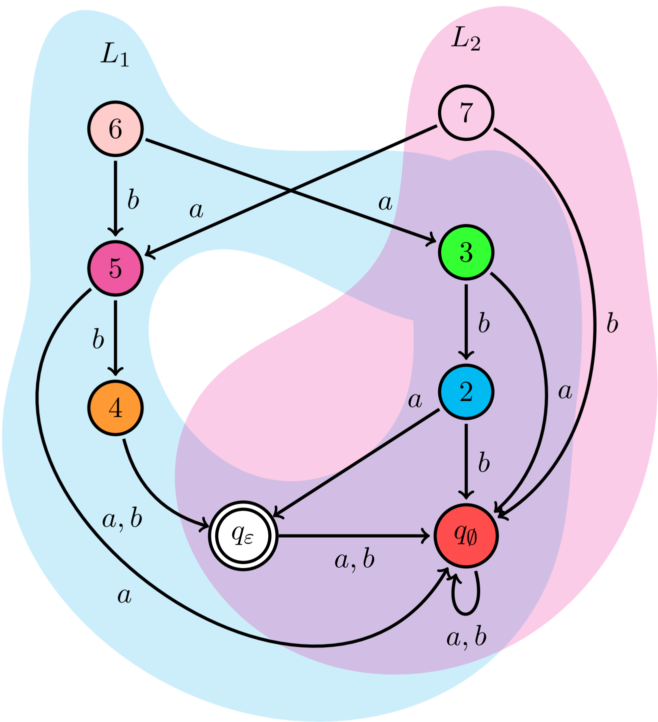
Executing `add-lang`(L_2) yields the following computation tree:



The table obtained after the execution is as follows:

Ident.	a-succ	b-succ
7	5	$q_\emptyset$
4	q_ε	q_ε
5	$q_\emptyset$	4

The resulting master automaton fragment is:

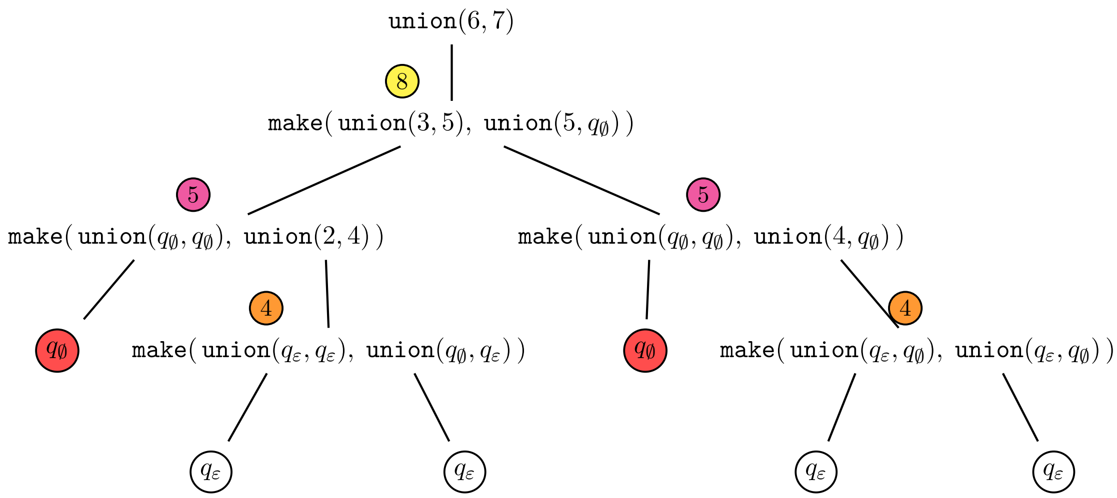


(c) Let us first adapt the algorithm for intersection to obtain an algorithm for union:

Input: States p and q of same length of the master automaton.
Output: State r of the master automaton such that $L(r) = L(p) \cup L(q)$.

`union`(p, q) *:*
    if $G(p, q)$ is not empty **then**
        return $G(p, q)$
    else if $p = q_\emptyset$ and $q = q_\emptyset$ **then**
        return $q_\emptyset$
    else if $p = q_\varepsilon$ or $q = q_\varepsilon$ **then**
        return q_ε
    else
        for $a_i \in \Sigma$ **do**
            $s_i \leftarrow$ `union`(p^{a_i}, q^{a_i})
        $G(p, q) \leftarrow$ `make`$(s_1, s_2, \ldots, s_n)$
        return $G(p, q)$

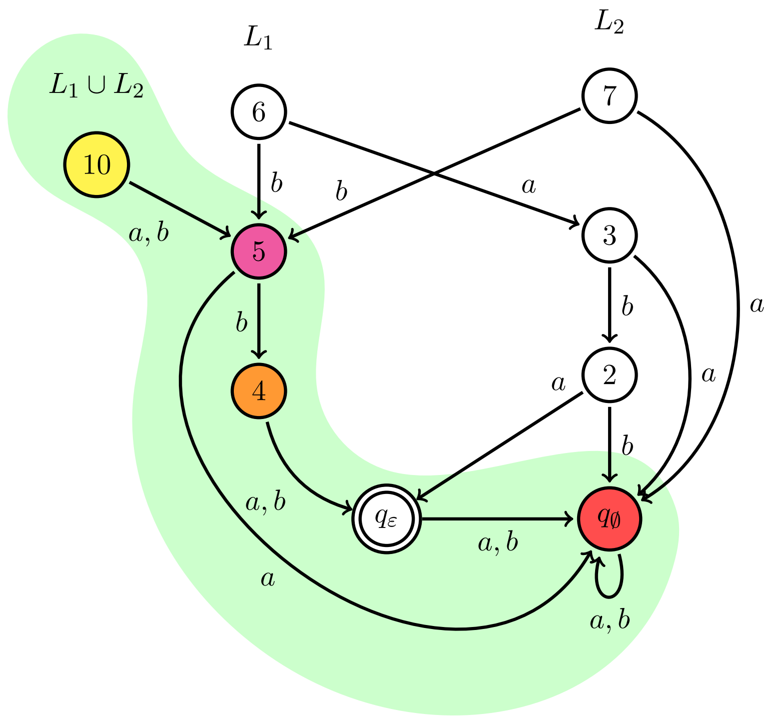
Executing `union`$(6, 7)$ yields the following computation tree:



The table obtained after the execution is as follows:

Ident.	a-succ	b-succ
8	5	5
5	$q_\emptyset$	4
4	q_ε	q_ε

The new fragment of the master automaton is:



[hard] Note that **union** could be slightly improved by returning q whenever $p = q$, and by updating $G(q, p)$ at the same time as $G(p, q)$.

(d) The kernels are:

$$\begin{aligned} \langle L_1 \rangle &= L_1, \\ \langle L_2 \rangle &= L_2, \\ \langle L_1 \cup L_2 \rangle &= \{ba, bb\}. \end{aligned}$$

Exercise 7.3.

We define the *language* of a Boolean formula φ over variables $x_1, \ldots, x_n$ as:

$$\mathcal{L}(\varphi) = \{a_1 a_2 \cdots a_n \in \{0,1\}^n : \text{ the assignment } x_1 \mapsto a_1, \ldots, x_n \mapsto a_n \text{ satisfies } \varphi\}.$$

(a) Give a polynomial-time algorithm that takes as input a DFA A recognizing a language of length n, and returns a Boolean formula φ such that $\mathcal{L}(\varphi) = \mathcal{L}(A)$.

(b) Give an exponential-time algorithm that takes a Boolean formula φ as input, and returns a DFA A recognizing $\mathcal{L}(\varphi)$.

Solution.

(a) The algorithm takes as input a state of the master automaton and the length of the language it recognizes, and recursively constructs a formula as follows:

Input: state q recognizing a language of length n
Output: formula φ_q such that $\mathcal{L}(\varphi_q) = \mathcal{L}(q)$

1 `DFAtoFormula(`q, n`)`*:*
2   **if** $G(q)$ is not empty **then**
3   **return** $G(q)$
4   **if** $q = q_\emptyset$ **then**
5   **return false**
6   **else if** $q = q_\epsilon$ **then**
7   **return true**
8   **else**
9         $\varphi_0 \leftarrow \mathit{DFAtoFormula}(q^0, n-1)$
10         $\varphi_1 \leftarrow \mathit{DFAtoFormula}(q^1, n-1)$
11         $\varphi_q \leftarrow (\neg x_1 \wedge \varphi_0) \vee (x_1 \wedge \varphi_1)$
12         $G(q) \leftarrow \varphi_q$
13   **return** $G(q)$

Observe that the parameter n is needed to identify the variable at line 11.

Our algorithm takes as input a table with the state identifiers and successors of all the descendants of q (i.e., the fragment of the master automaton starting at q). This is a polynomial time algorithm because we compute $\varphi_{q'}$ once for every descendant q' of q.

Note that this algorithm could be improved by adding an *else* that checks if $q^0 = q^1$ before the last else:

else if $q^0 = q^1$ **then**
  $\varphi \leftarrow \mathit{DFAtoFormula}(q^0, n-1)$
  $\varphi_q \leftarrow \varphi$
  $G(q) \leftarrow \varphi_q$
  return $G(q)$

(b) Given a formula φ over variables $x_1, \ldots, x_n$, we write $\varphi[x_i/\mathbf{true}]$ and $\varphi[x_i/\mathbf{false}]$ to denote the formulas obtained by replacing all occurrences of x_i in φ by **true** and **false**, respectively. We have that $\mathcal{L}(\varphi[x_1/\mathbf{false}]) = \mathcal{L}(\varphi)^0$ and $\mathcal{L}(\varphi[x_1/\mathbf{true}]) = \mathcal{L}(\varphi)^1$. This yields the following algorithm:

Input: formula φ over variables $x_1, \ldots, x_n$, total number of variables n, k initially equal to 1
Output: state q such that $L(\varphi) = L(q)$
`FormulatoDFA(`φ, n, k`)`:
  if $G(\varphi)$ is not empty **then**
    return $G(\varphi)$
  if $\varphi = \mathbf{true}$ **then**
    return q_ε
  else if $\varphi = \mathbf{false}$ **then**
    return $q_\emptyset$
  else
    $r_0 \leftarrow \mathit{FormulatoDFA}(\varphi[x_k/\mathbf{false}], n, k+1)$
    $r_1 \leftarrow \mathit{FormulatoDFA}(\varphi[x_k/\mathbf{true}], n, k+1)$
    $G(\varphi) \leftarrow \mathit{make}(r_0, r_1)$
    return $G(\varphi)$

Puzzle exercise 7.4.

Let $n \in \mathbb{N}$. Construct an NFA $N = (Q, \Sigma, \delta, Q_0, F)$ with $\mathcal{O}(n)$ states, s.t. the shortest word in $\Sigma^* \setminus L(N)$ has length at least 2^n.

Notes: The puzzle exercises cover advanced material and are not directly relevant to the exam. No model solutions will be provided. You can get feedback on your solution by sending it to me (Zulip or mail), or coming to my office in person (MI 03.11.037).

Solution (tutors only). We choose $\Sigma := \{0, 1, \#\}$. The goal is for N to accept the language

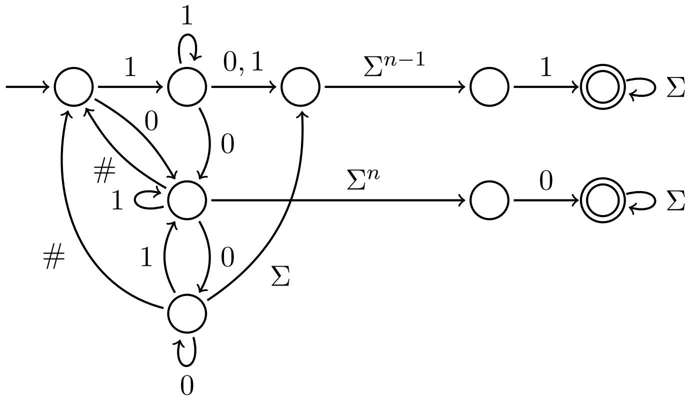
$$\Sigma^* \setminus \{w_0\#...\#w_l : w_0, ..., w_l \in \{0,1\}^n, (w_i)_2 = i, l = 2^n - 1\}$$

where $(w)_2$ denotes the value of w in least-significant bit first encoding. For example, $(0101)_2 = 5$. We first construct a DFA D for the language of the RE $r := 0^n(\#(0+1)^n)^*\#1^n$. Consider the set of regular expressions

$$\begin{aligned} R :=& \{0^i(\#(0+1)^n)^*\#1^n : i \in \{0, ..., n\}\} \\ &\cup \{(0+1)^i(\#(0+1)^n)^*\#1^n + 1^i : i \in \{0, ..., n\}\} \\ &\cup \{(0+1)^i(\#(0+1)^n)^*\#1^n : i \in \{1, ..., n\}\} \\ &\cup \{\emptyset\} \end{aligned}$$

Clearly, the languages generated by this set contain $L(r)$ and are closed under taking residual languages, so we can define $D := (R, \Sigma, \delta, r, \{r \in R : \varepsilon \in L(r)\})$, with $\delta(r, a) := r'$ for $a \in \Sigma$, where $r' \in R$ is some RE with $L(r') = L(r)^a$. Note that D has $3n \in \mathcal{O}(n)$ states.

We then construct an NFA N', which looks as follows:



The edges labelled with Σ^n and Σ^{n-1} must be replaced by a sequence of states, so this NFA has $\mathcal{O}(n)$ states in total. It accepts all words in the language $\{w_0\#...\#w_l : w_0, ..., w_l \in \{0,1\}^n \wedge \exists i.(w_i)_2 \neq (w_{i+1})_2\}$.

Finally, we obtain N s.t. $L(N) = \overline{L(D)} \cup L(N')$, which can be done by putting N' and the complement of D "side-by-side", so it has $\mathcal{O}(n)$ states as well.