Prof. Javier Esparza
Philipp Czerner

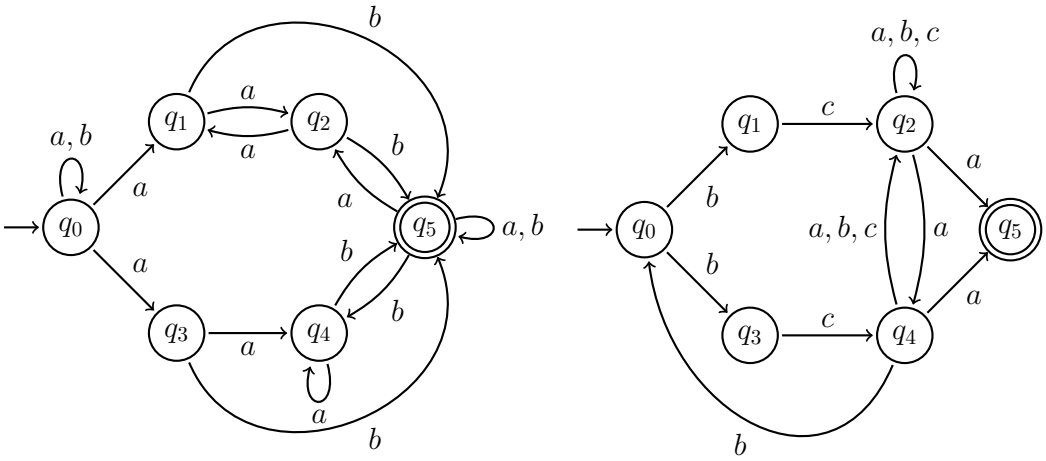
Technical University of Munich
Chair for Foundations of Software Reliability

Automata and Formal Languages

Winter Term 2023/24 – Exercise Sheet 4

Exercise 4.1.

Let A and B be respectively the following NFAs:



(a) Compute the coarsest stable refinements (CSR) of A and B.

(b) Construct the quotients of A and B with respect to their CSRs.

(c) Show that

$$L(A) = \{w \in \{a,b\}^* : w \text{ contains an occurrence of the subword } ab\}$$
$$L(B) = \{w \in \{a,b,c\}^* : w \text{ starts with } bc \text{ and ends with } a\}$$

(d) Are the automata obtained in (b) minimal?

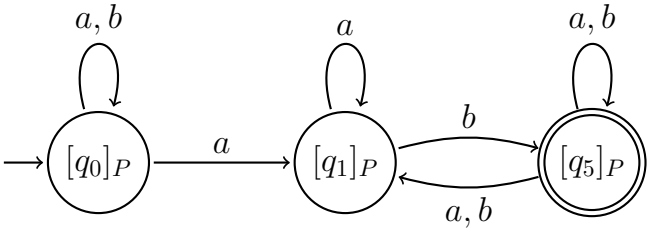
Solution.

(1) (a)

Iter.	Block to split	Splitter	New partition
0	—	—	$\{q_0, q_1, q_2, q_3, q_4\}, \{q_5\}$
1	$\{q_0, q_1, q_2, q_3, q_4\}$	$(b, \{q_5\})$	$\{q_0\}, \{q_1, q_2, q_3, q_4\}, \{q_5\}$
2	none, partition is stable	—	—

The CSR is $P = \{\{q_0\}, \{q_1, q_2, q_3, q_4\}, \{q_5\}\}$.

(b)



(c) The automaton A and the automaton obtained from (b) accept the same language. Notice that in the automaton from (b), there is an accepting run for a word w which visits the final state exactly once if and only if $w \in \Sigma^* ab$. Since there are self-loops at the final state for both a and b, it follows that the language of this automaton is $\Sigma^* ab \Sigma^*$.

(d) Yes. By (c), the language accepted by A is $\Sigma^* ab \Sigma^*$. An NFA with one state can only accept $\emptyset, \{\varepsilon\}, a^*, b^*$ and $\{a, b\}^*$. Suppose there exists an NFA $A' = (\{q_0, q_1\}, \{a, b\}, \delta, Q_0, F)$ accepting $L(A)$. Without loss of generality, we may assume that q_0 is initial. A' must respect the following properties:

- $q_0 \notin F$, since $\varepsilon \notin L(A)$,
- $q_1 \in F$, since $L(A) \neq \emptyset$,
- $q_1 \notin Q_0$, since $\varepsilon \notin L(A)$,
- $\delta(q_0, a)$ is non-empty, otherwise it is impossible to accept ab. Further, $q_1 \notin \delta(q_0, a)$, otherwise it is possible to accept a. Hence, $\delta(q_0, a) = \{q_0\}$.
- $q_1 \in \delta(q_0, b)$, otherwise it is impossible to accept ab.

This implies that A' accepts b, yet $b \notin L(A)$. Therefore, no NFA with two states can accept $L(A)$. □

(2) (a)

Iter.	**Block to split**	**Splitter**	**New partition**
0	—	—	$\{q_0, q_1, q_2, q_3, q_4\}, \{q_5\}$
1	$\{q_0, q_1, q_2, q_3, q_4\}$	$(a, \{q_5\})$	$\{q_0, q_1, q_3\}, \{q_2, q_4\}, \{q_5\}$
2	$\{q_2, q_4\}$	$(b, \{q_0, q_1, q_3\})$	$\{q_0, q_1, q_3\}, \{q_2\}, \{q_4\}, \{q_5\}$
3	$\{q_0, q_1, q_3\}$	$(c, \{q_4\})$	$\{q_0, q_1\}, \{q_3\}, \{q_2\}, \{q_4\}, \{q_5\}$
4	$\{q_0, q_1\}$	$(c, \{q_2\})$	$\{q_0\}, \{q_1\}, \{q_3\}, \{q_2\}, \{q_4\}, \{q_5\}$

The CSR is $P = \{\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}, \{q_5\}\}$.

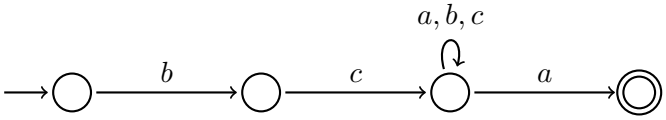
(b) The automaton remains unchanged.

(c) $\supseteq$) Suppose w starts with bc and ends with a. If $w = w_1 w_2 ... w_n$, then $q_0, q_1, \underbrace{..., q_2, ...}_{n-3 \text{ times}}, q_5$ is a valid accepting run for w.

$\subseteq$) Let $w \in L(B)$. Note that every outgoing edge from q_0 is labelled by a b and goes to either q_1 or q_3 and every outgoing edge from both q_1 and q_3 is labelled by a c. It follows that any path from q_0 to q_5 must involve reading a bc at the beginning. Further, all the incoming edges to q_5 are labelled by an a. It follows that any path from q_0 to q_5 must involve reading an a at the

end. Since $w \in L(B)$, it then follows that w must begin with bc and end with a. □

(d) No. The following NFA with four states accepts the same language.



Exercise 4.2.

Let $\Sigma = \{a, b\}$. For any $n \in \mathbb{N}$, let $L_n := \{ww^R : w \in \Sigma^n\}$, where w^R is the reverse of w, e.g. $(abc)^R = cba$. It is known that every NFA (and hence also every DFA) recognizing L_n must have at least 2^n states. We refine this bound here for DFAs.

(a) Construct A_2, the minimal DFA for L_2.

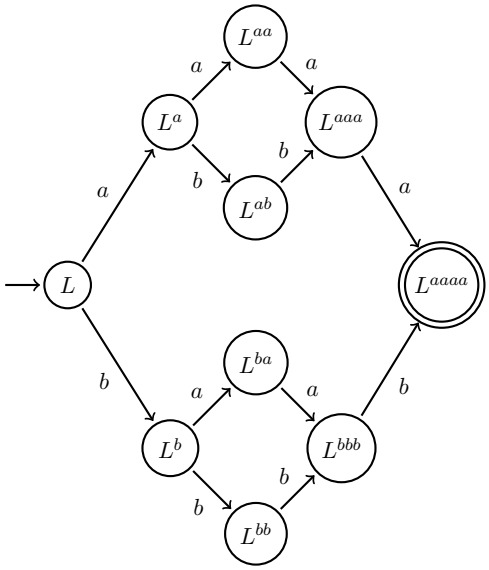
(b) What are the residuals of L_2? Assign them to the states of the DFA you gave for (a).

(c) Give a construction for a DFA that accepts L_n.

(d) How many states does the minimal DFA for L_n contain, for $n \geq 2$?

Solution.

(a) The trap state is omitted for the sake of readability:



(b) We have $L_2 = \{aaaa, abba, baab, bbbb\}$. We compute the residuals L^w for all words w by increasing length of w.

- $|w| = 0$: $L^\varepsilon = \{aaaa, abba, baab, bbbb\}$.
- $|w| = 1$: $L^a = \{aaa, bba\}$ and $L^b = \{aab, bbb\}$.
- $|w| = 2$: $L^{aa} = \{aa\}$, $L^{ab} = \{ba\}$, $L^{ba} = \{ab\}$ and $L^{bb} = \{bb\}$.
- $|w| = 3$: $L^{aaa} = \{a\} = L^{abb}$, and $L^{baa} = \{b\} = L^{bbb}$.
- $|w| \geq 4$: $L^w = \begin{cases} \{\varepsilon\} & \text{if } w \in L_k, \\ \emptyset & \text{otherwise.} \end{cases}$

(c) Notice that L_{k+1} is simply $aL_ka + bL_kb$ for any $k \geq 2$. Using this observation, we generalize the construction given in (a) for $k = 2$, by induction on k. The base case of $k = 2$ has been done already. Suppose we have already constructed $A_k = (Q_k, \{a, b, \}, \delta_k, q_0^k, q_f^k)$ with the property that it has exactly one initial state, one final state and one trap state $trap_k$ (Note that A_2 satisfies this property). We now construct $A_{k+1} = (Q_{k+1}, \{a, b, \}, \delta_{k+1}, q_0^{k+1}, q_f^{k+1})$ as follows:

The set of states Q_{k+1} is taken to be $\{q_0^{k+1}, q_f^{k+1}, trap_{k+1}\} \cup ((Q_k \backslash \{trap_k\}) \times \{1, 2\})$, where $q_0^{k+1}, q_f^{k+1}, trap_{k+1}$ are three fresh states. Intuitively we add a fresh initial state, a fresh final state, a fresh trap state and take two *copies* of the states of A_k while removing $trap_k$.

The transition function δ_{k+1} is defined as follows:

- $\delta_{k+1}(q_0^{k+1}, a) = (q_0^k, 1)$ and $\delta_{k+1}(q_0^{k+1}, b) = (q_0^k, 2)$. Intuitively, upon reading an a (resp. b) from the initial state of A_{k+1}, we move to the initial state of the first (resp. second) copy of A_k.
- $\delta_{k+1}(q_f^k, a) = q_f^{k+1}$ and $\delta_{k+1}(q_f^k, b) = q_f^{k+1}$. Intuitively, upon reading an a (resp. b) from the final state of the first (resp. second) copy of A_{k+1}, we move to the final state of of A_{k+1}.
- $\delta_{k+1}((q, i), a) = p$ where $p = (\delta_k(q, a), i)$ if $\delta_k(q, a) \neq trap_k$ and otherwise $p = trap_{k+1}$. Intuitively, within a copy of A_k, we follow the transitions of A_k and stay within that copy itself if the state that we are supposed to go to is not the trap state of A_k. Otherwise, instead of going to the trap state of A_k, we go to the trap state of A_{k+1}.

Assuming that A_k recognizes L_k, we can then show that A_{k+1} recognizes L_{k+1}. By induction, this will show that our construction is correct.

(d) Note that if $f(k)$ is the number of states that A_k has, (where A_k is the DFA defined in the previous subproblem), then $f(2) = 11$ and $f(k+1) = 2(f(k) - 1) + 3 = 2f(k) + 1$. Solving this, we get $f(k) = 3 \cdot 2^k - 1$. We claim that A_k is a minimal DFA, by induction on k. The base case of $k = 2$ is already done. For the induction step, suppose p, q are two distinct states of A_{k+1}. We will show that $L_{A_{k+1}}(p) \neq L_{A_{k+1}}(q)$.

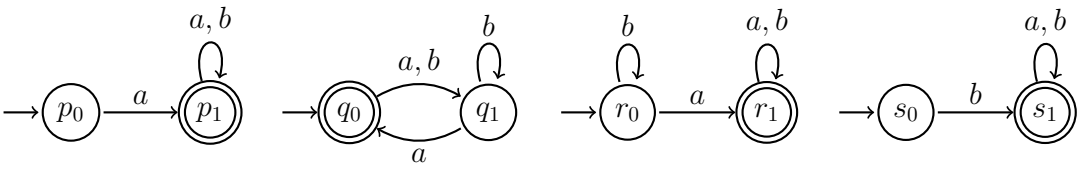
Notice that the initial state q_0^{k+1} recognizes only strings of length $2k + 2$ and the final state q_f^{k+1} recognizes only ϵ, whereas the other states of A_{k+1} do not recognize any of these strings. This implies that the languages of the initial and the final states are different from the rest. Similarly, the language of the trap state is also different from the rest.

Hence, we can assume that $p = (p', i)$ and $q = (q', j)$ for some $p', q' \in Q_k$ and some $i, j \in \{1, 2\}$. If $i \neq j$, then p and q belong to different copies of A_k. Let $i = 1$ and $j = 2$. Notice that $L_{A_{k+1}}(p) = L_{A_k}(p')a$ and $L_{A_{k+1}}(q) = L_{A_k}(q')b$. Hence $L_{A_{k+1}}(p) \neq L_{A_{k+1}}(q)$.

The only case left is when $i = j$. In this case notice that $L_{A_{k+1}}(p) = L_{A_k}(p')c$ and $L_{A_{k+1}}(q) = L_{A_k}(q')c$ where c is either a or b, depending on whether i is 1 or 2. By induction hypothesis, A_k is the minimal DFA for L_k and so $L_{A_k}(p')$ and $L_{A_k}(q')$ are different. Hence $L_{A_{k+1}}(p) \neq L_{A_{k+1}}(q)$, thereby concluding the proof.

Exercise 4.3.

Consider the following DFAs A, B, C and D:

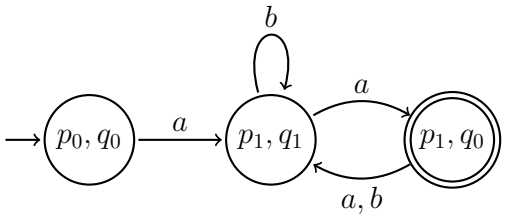


(a) Use pairings to decide *algorithmically* whether $L(A) \cap L(B) \subseteq L(C)$.

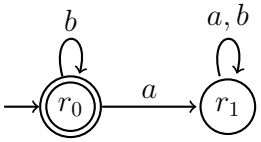
(b) Use pairings to decide *algorithmically* whether $L(D) \subseteq L(A) \cap L(B)$.

Solution.

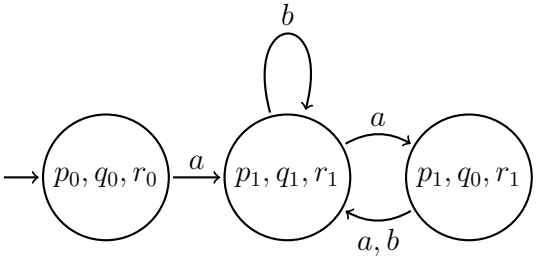
(a) We first build the pairing accepting $L(A) \cap L(B)$. Note that it is not necessary to explore the implicit trap states of A and B as they cannot lead to final states in the pairing. We obtain:



Now, we build the pairing accepting $(L(A) \cap L(B)) \setminus L(C)$, or equivalently $(L(A) \cap L(B)) \cap \overline{L(C)}$, from the above automaton and C. Recall that the complement of C is the following automaton:



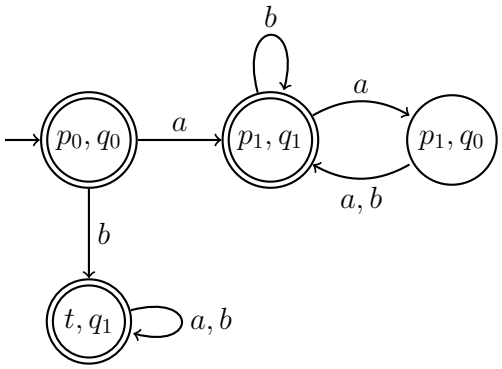
Once again, it is not necessary to explore the implicit trap states of the automaton for $L(A) \cap L(B)$. The following automaton is the pairing accepting $(L(A) \cap L(B)) \cap \overline{L(C)}$:



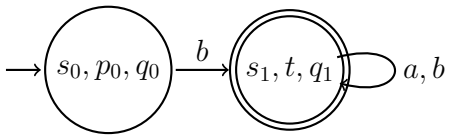
Since the above automaton does not contain final states, its language is empty and hence $L(A) \cap L(B) \subseteq L(C)$.

(b) This time we want to check whether $L(D) \setminus (L(A) \cap L(B))$ is empty. That is, we need to construct the pairing $L(D) \cap \overline{(L(A) \cap L(B))}$. Thus, it is not necessary to explore the implicit trap states of the automaton D, but it is necessary for A and

B, as their trap states may be part of final states in the pairing. First we obtain the automaton accepting $\overline{(L(A) \cap L(B))}$:



Now, we build the pairing accepting $L(D) \cap \overline{(L(A) \cap L(B))}$. We obtain:



Since the above automaton contains a final state, it means that there is a word in the language $L(D) \setminus (L(A) \cap L(B))$, that is, there is a word accepted by D, but not by A and B. For example, any word starting with a letter b. Therefore, it is not true that $L(D) \subseteq L(A) \cap L(B)$.

Exercise 4.4.

In this exercise we consider four (AI-generated) proof attempts. For each proof attempt, determine whether the proof is correct and mark the errors it contains (if any).

Claim 1. *Let X denote a language. Then X^* is regular.*

Proof attempt. To prove that the Kleene closure (star) of a language X, denoted as X^*, is regular, we can use a straightforward approach. If X is any language, and we take its Kleene closure X^*, then X^* is regular. This proof is based on the properties of regular languages and closure properties.

- **Basis:** The basis for the proof lies in the definition of regular languages. The empty set $\emptyset$ and the set containing the empty string $\{\varepsilon\}$ are both regular. Any single character from the alphabet, say $\{a\}$, is also regular.
- **Inductive Step:** Now, we can use closure properties of regular languages to show that if X is regular, then X^* is regular. The closure properties state that regular languages are closed under certain operations, including union, concatenation, and Kleene closure.
  - **Union:** If L_1 and L_2 are regular languages, then their union $L_1 \cup L_2$ is also regular.
  - **Concatenation:** If L_1 and L_2 are regular languages, then their concatenation $L_1 L_2$ is also regular.
  - **Kleene Closure:** If L is a regular language, then L^* is also regular.

Since X is any language, it can be empty or non-empty, finite or infinite, regular or non-regular. It doesn't matter because X^* is defined as the set of all possible strings that can be formed by concatenating zero or more strings from X.

So, by the closure property of the Kleene closure operation, X^* is regular.

In summary, regardless of whether X is regular or not, the Kleene closure X^* is always regular because regular languages are closed under the Kleene closure operation. □

Claim 2. *Let X denote a language. Then X^* is not regular in general. (In other words: There exists an X s.t. X^* is not regular.)*

Proof attempt. To disprove that the Kleene closure (star) of a language X, denoted as X^*, is not necessarily regular, we can use a counterexample. There exist languages for which their Kleene closure is not regular. One classic example is the language $L = \{a^n b^n : n \geq 0\}$, which is well-known to be non-regular. Let's show that L^* is also non-regular:

- **Base Case:** L contains the strings $\varepsilon, ab, aabb, aaabbb, ...$
- **Inductive Step:** Now, let's consider L^*, the Kleene closure of L. It consists of all possible strings that can be formed by concatenating zero or more strings from L. L^* includes strings like $\varepsilon, ab, aaabbb, aabbabbb$, etc., which can have any number of 'a's and 'b's but must have an equal number of 'a's and 'b's. This is because L itself is the set of strings with an equal number of 'a's and 'b's.

L^* contains not only the strings in L but also strings that do not have an equal number of 'a's and 'b's, and this set of strings is not regular. For example, the string $a^p b^q$, where $p \neq q$, is not in L and is not regular. If you concatenate such strings in L^*, you will obtain non-regular strings.

Therefore, L^* is not regular because it contains non-regular strings. Since L^* is a subset of X^* (where X is any language), this demonstrates that X^* is not necessarily regular. □

Claim 3. *For a word w with length divisible by 3, let $f(w)$ denote the word you obtain by deleting the middle third. For example $f(aba) = aa$ and $f(abbabaaab) = abbaab$. Let L denote a regular language. Then $f(L)$ is regular.*

Proof attempt. To prove that the language $f(L)$ is regular for a given regular language L, we can construct a finite automaton that recognizes it. We can use the fact that regular languages are closed under certain operations, including the operation of deleting a contiguous portion of a string, as long as it is done consistently.

Let's define a finite automaton to recognize $f(L)$. This automaton will have states that represent the possible "states" of the string as we process it.

The automaton will work as follows:

(a) Start in the initial state.

(b) For each symbol in the input string, the automaton will transition to a new state. Initially, the automaton stores the characters in the first third of the string. As it reads more symbols, it continues to build the first third.

(c) When it reaches the last third of the input string, it transitions to an accepting state, indicating that the entire first two-thirds of the string have been processed.

(d) Any remaining characters in the last third of the input string are ignored, as they are not part of $f(w)$.

The automaton will have a state for each possible combination of characters in the first third of the string. Since L is regular, there is a finite number of such combinations, and the automaton can have a finite number of states.

Since we can construct a finite automaton to recognize $f(L)$, we can conclude that $f(L)$ is regular. This is based on the closure property of regular languages under the operation of deleting a contiguous portion of a string while processing it. □

Claim 4. *For a word w with length divisible by 3, let $f(w)$ denote the word you obtain by deleting the middle third. For example $f(aba) = aa$ and $f(abbabaaab) = abbaab$. Let L denote a regular language. Then $f(L)$ is not regular in general.*

Proof attempt. To disprove that the language $f(L)$ is necessarily regular for a regular language L, we can provide a counterexample. We will show that there exists a regular language L for which $f(L)$ is not regular.

Consider the regular language $L = \{a^{3n} : n \geq 0\}$, which consists of strings composed of a multiple of 3 'a' symbols. This language is regular because it can be recognized by a simple finite automaton that checks for strings of the form a^{3n}.

Now, let's look at the language $f(L)$. This language contains words obtained by removing the middle third of strings in L. The middle third contains 'n' 'a' symbols. Therefore, $f(L)$ consists of strings of the form a^{2n} for $n \geq 0$, which means it includes all strings with an even number of 'a' symbols.

However, the set of all strings with an even number of 'a' symbols is not regular. This can be shown using the pumping lemma for regular languages. If you assume that $f(L)$ is regular, then you would expect to be able to apply the pumping lemma to it, but it fails for such strings. You can't divide these strings into substrings to satisfy the pumping lemma's conditions because, after pumping, the number of 'a's in the string will still be even, which violates the requirement for the language to be regular.

Therefore, we've demonstrated that there exists a regular language L for which $f(L)$ is not regular, which disproves the claim that $f(L)$ is necessarily regular for all regular languages L. □

Solution. The four proofs above are all wrong.

(1) The first problem is starting an induction in line 5, without specifying what exactly the parameters of the induction are. There is no mention of induction hypothesis, and neither is it clear how one would go about proving Claim 1 with induction. Overall, the whole induction in lines 5-16 is just a list of true statements, but they are not connected to the actual statement the proof is trying to show.

There is also a minor mistake in line 7, it should read "any *set* containing a single character from the alphabet is regular".

Lines 17-19 simply restate that the claim is true, without providing justification. In lines 20-22 there is the argument that the claim follows from regular languages being closed under the Kleene start operation, which is nonsensical, as the closure property applies only to regular languages.

(2) Again, the induction in line 6 lacks context. It is not clear how to use induction to argue that a language is not regular. There is a small error in line 9 as $aabbabbb \notin$

L^*. In lines 10-11 the proof states that L is the set of strings with an equal number of a and b, which is false; L is only a subset of that language. Additionally, it is unclear in what sense this is an induction step.

Then in line 12 it is stated that L^* contains words that have an unequal number of a and b, which it does not. The statement in line 13 of "this set of strings is not regular" is ambiguous, but assuming it applies to the language $\{w \in \Sigma^* : |w|_a \neq |w|_b\}$ it holds.

Then lines 13-15 state that particular words are not regular, which is nonsensical (words cannot be irregular, only languages). This also invalidates the conclusion in line 16. Finally, line 17 is obviously wrong as well, as $L^* \subseteq X^*$ does not hold for all X.

(3) Lines 2-4 are too vague (what does "consistently" mean) and lack justification. The description in lines 7-15 is confused in multiple ways. Line 9 is imprecise. Line 10 assumes infinitely many states. Line 12 does not indicate how the automaton is supposed to determine that it has reached the last third. Line 14 makes it seem as though the last third is deleted, but it is supposed to be the middle third.

Lines 17-18 argue why it is possible to store the first third of the word using only finitely many states, but the argument does not convince. The connection with a language being regular is unclear. Lines 20-21 reference a closure property that does not exist.

(4) Lines 5-6 argue in circular fashion. Apart from this, lines 1-10 are (surprisingly) correctly reasoned. However, line 11 is wrong. Lines 14-16 do not provide justification (and are also arguing a falsity).