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Automata and Formal Languages

Winter Term 2023/24 – Exercise Sheet 2

Exercise 2.1.

Consider the regular expression $r = (a + ab)^*$.

(a) Convert r into an equivalent NFA-ε A.

(b) Convert A into an equivalent NFA B. (It is not necessary to use algorithm $NFA\varepsilon toNFA$)

(c) Convert B into an equivalent DFA C.

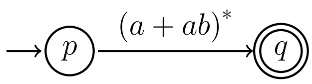
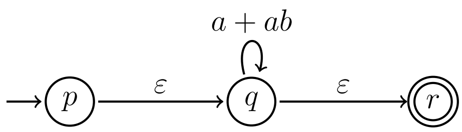
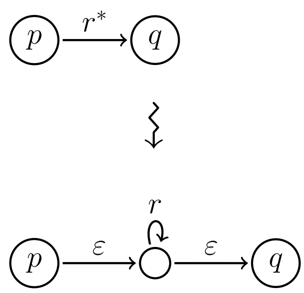
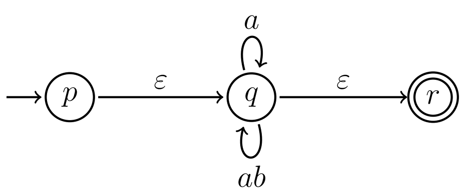
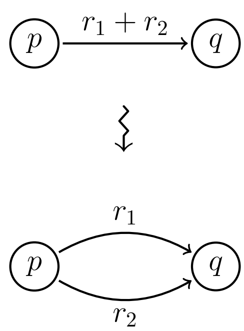
(d) By inspecting B, give an equivalent minimal DFA D. (No algorithm needed).

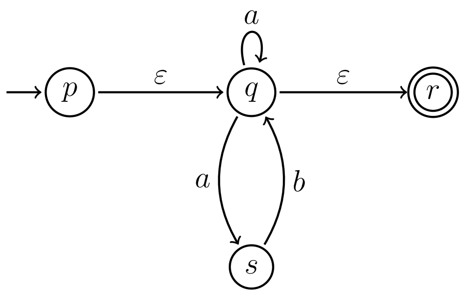
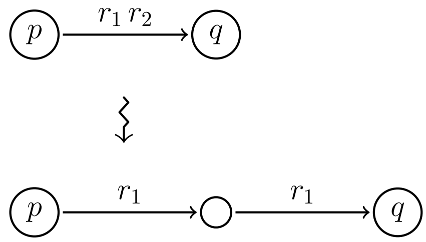
(e) Convert D into an equivalent regular expression r'.

(f) Prove formally that $L(r) = L(r')$.

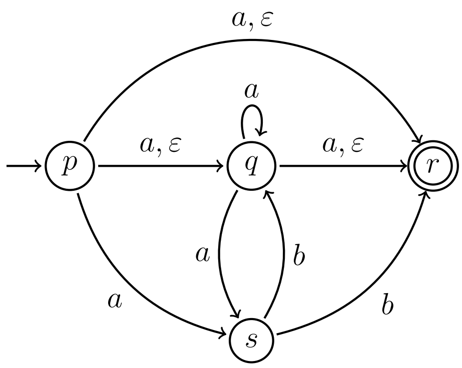
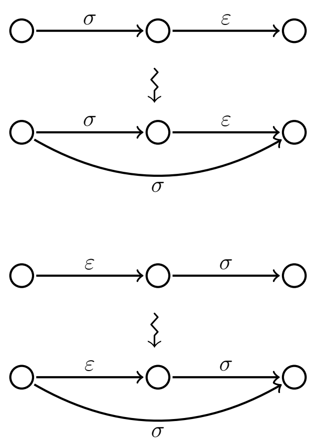
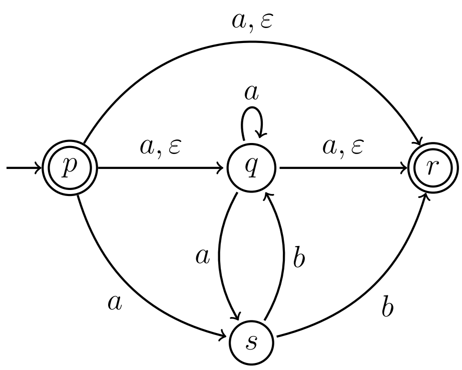
Solution.

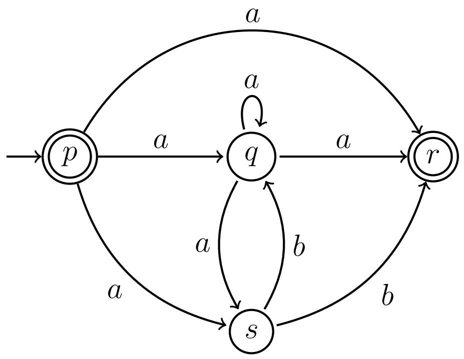
(a)

Iter.	Automaton obtained	Rule applied
1		Initial automaton from reg. expr.
2		
3		

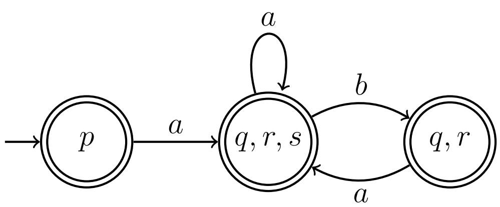
4		

(b)

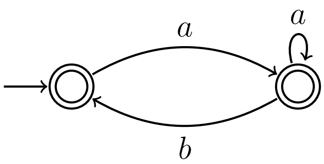
Iter.	**Automaton obtained**	**Rule applied**
1		 where $\sigma \in \Sigma \cup \{\varepsilon\}$
2		Initial states that can reach a final state through ε-transitions are made final.

3		Remove ε-transitions. Remove states non reachable from initial state.

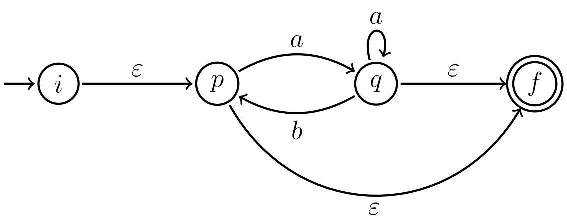
(c)

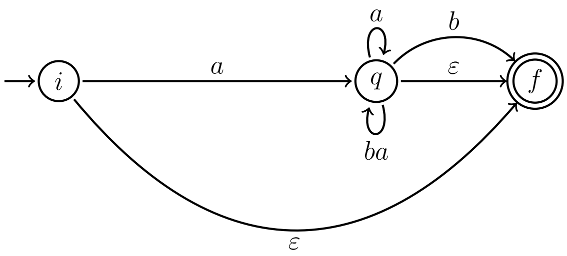
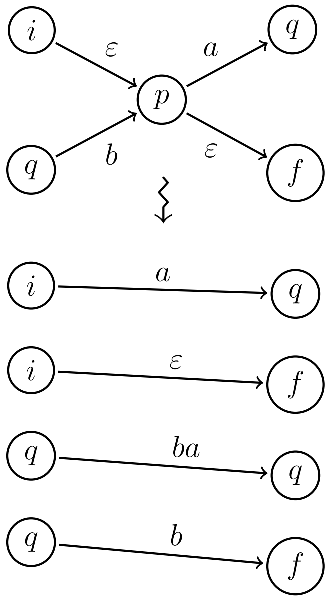
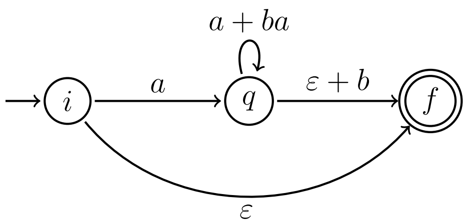
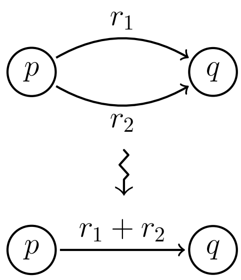
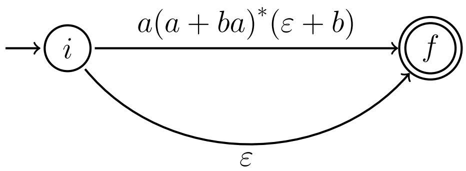
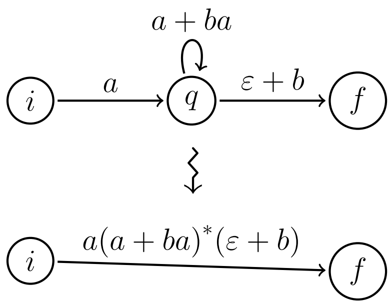
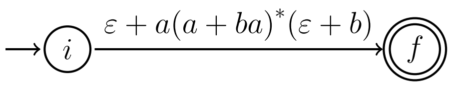
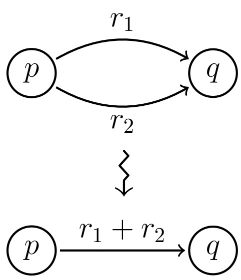


(d) States $\{p\}$ and $\{q, r\}$ have the exact same behaviours, so we can merge them. Indeed, both states are final and $\delta(\{p\}, \sigma) = \delta(\{q, r\}), \sigma)$ for every $\sigma \in \{a, b\}$. We obtain:



(e)

Iter.	**Automaton obtained**	**Rule applied**
1		Add single initial and final states.

2		
3		
4		
5		
6	$\varepsilon + a(a+ba)^*(\varepsilon + b)$	Extract regular expression from the unique transition.

(f) Let us first show that $a(a+ba)^i = (a+ab)^i a$ for every $i \in \mathbb{N}$. We proceed by induction on i. If $i = 0$, then the claim trivially holds. Let $i > 0$. Assume the claims holds at $i-1$. We have

$$\begin{aligned} a(a+ba)^i &= a(a+ba)^{i-1}(a+ba) & \\ &= (a+ab)^{i-1}a(a+ba) & \text{(by induction hypothesis)} \\ &= (a+ab)^{i-1}(aa+aba) & \text{(by distributivity)} \\ &= (a+ab)^{i-1}(a+ab)a & \text{(by distributivity)} \\ &= (a+ab)^i a. & \end{aligned}$$

This implies that

$$a(a+ba)^* = (a+ab)^* a. \tag{1}$$

We may now prove the equivalence of the two regular expressions:

$$\begin{aligned} \varepsilon + a(a+ba)^*(\varepsilon+b) &= \varepsilon + (a+ab)^* a(\varepsilon+b) & \text{(by (1))} \\ &= \varepsilon + (a+ab)^*(a+ab) & \text{(by distributivity)} \\ &= \varepsilon + (a+ab)^+ & \\ &= (a+ab)^*. & \end{aligned}$$

□

Exercise 2.2.

Prove or disprove the following.

(a) If L_1 and $L_1 \cup L_2$ are regular, then L_2 is regular.
(b) If L_1 and $L_1 \cap L_2$ are regular, then L_2 is regular.
(c) If L_1 and $L_1 L_2$ are regular, then L_2 is regular.
(d) If L^* is regular, then L is regular.

Solution. All of these claims are false. Let $\Sigma = \{a\}$. Note that since there are an uncountable number of languages over Σ which contain the words ϵ and a, but only a countable number of DFAs, it follows that there must be a non-regular language L' such that $\epsilon, a \in L'$.

(a) Let $L_1 = \Sigma^*$ and $L_2 = L'$. Since $L_1 \cup L_2 = \Sigma^*$, the claim is false.
(b) Let $L_1 = \emptyset$ and $L_2 = L'$. Since $L_1 \cap L_2 = \emptyset$, the claim is false.
(c) Let $L_1 = \Sigma^*$ and $L_2 = L'$. Since $\epsilon \in L'$, it follows that $L_1 L_2 = \Sigma^*$ and so the claim is false.
(d) Let $L = L'$. Since $a \in L'$, it follows that $L^* = \Sigma^*$ and so the claim is false.

Exercise 2.3.

Recall that a nondeterministic automaton A accepts a word w if at least one of the runs of A on w is accepting. This is sometimes called the *existential* accepting condition. Consider the variant in which A accepts w if *all* runs of A on w are accepting (in particular, if A has no run on w then it accepts w). This is called the *universal* accepting condition and such automata will be referred to as a co-NFA.

Intuitively, we can visualize a co-NFA as executing all runs in parallel. After reading a word w, the automaton is simultaneously in all states reached by all runs labelled by w, and accepts if all those states are accepting.

(a) Suppose A_1 and A_2 are two co-NFA which accept languages L_1 and L_2 respectively. Let n_1 and n_2 be the number of states of A_1 and A_2 respectively. Show that there is a co-NFA B over $n_1 + n_2$ states which accepts $L_1 \cap L_2$.

(b) Give an algorithm that transforms a co-NFA into a DFA recognizing the same language. This shows that automata with universal accepting condition recognize the regular languages.

Let $\Sigma = \{a, b\}$. Given a word $w = a_1 a_2 ... a_n$ where each $a_i \in \Sigma$, let $w^R = a_n a_{n-1} ... a_1$ denote the *reverse* of w. For any $n \in \mathbb{N}$, consider the language $L_n := \{ww^R \in \Sigma^{2n} \mid w \in \Sigma^n\}$.

(c) Give a co-NFA with $O(n^2)$ states that recognizes L_n.

(d) Prove that every NFA (and hence also every DFA) recognizing L_n has at least 2^n states.

Solution.

(a) Let $A_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$ be the given two co-NFAs. Let B be the co-NFA given by $B = (Q_1 \cup Q_2, \Sigma, \delta_1 \cup \delta_2, I_1 \cup I_2, F_1 \cup F_2)$. Notice that if $|Q_1| = n_1$ and $|Q_2| = n_2$, then the number of states of B is $n_1 + n_2$. Further, note that ρ is a run of B on a word w if and only if ρ is either a run of A_1 on w or ρ is a run of A_2 on w. It follows that all runs of B on a word w are accepting if and only if all runs of A_1 and A_2 on w are accepting. Hence, B accepts $L_1 \cap L_2$.

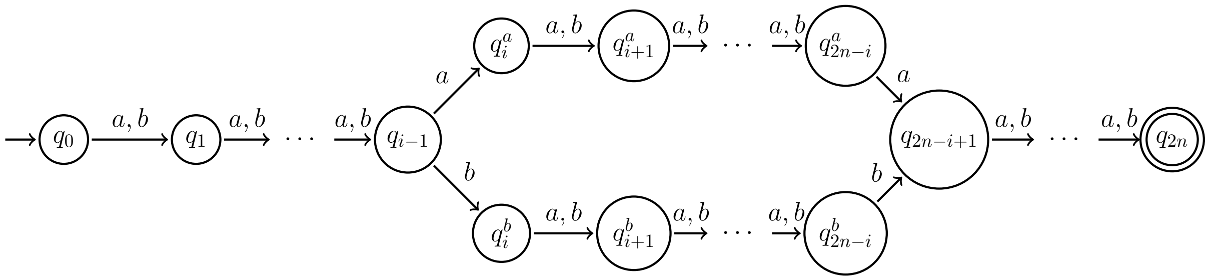
(b) Let $A = (Q, \Sigma, \delta, Q_0, F)$ be a co-NFA. We do the same powerset construction that we do for NFAs to get a DFA $B = (\mathcal{Q}, \Sigma, \Delta, q_0, \mathcal{F})$ except we now set $\mathcal{F} = \{Q' \in \mathcal{Q} : Q' \subseteq F\}$. All the other elements are defined in exactly the same way as is done for the powerset construction.

(c) For any $n \in \mathbb{N}$ and any $1 \leq i \leq n$, let

$$L_n^i := \{w : w \in \Sigma^{2n}, \text{ the } i^{th} \text{ letter of } w \text{ and the } (2n-i+1)^{th} \text{ letter of } w \text{ are the same } \}$$

Notice that $L_n = \bigcap_{1 \leq i \leq n} L_n^i$. By a), it follows that if we give a co-NFA of size $O(n)$ for each L_n^i, then we have a co-NFA of size $O(n^2)$ for L_n.

We now construct a co-NFA of size $O(n)$ for each L_n^i, as given by the following illustration.



First, the automaton has a sequence of states $q_0, q_1, ..., q_{i-1}$ with transitions $q_j \xrightarrow{a,b} q_{j+1}$ for every $0 \leq j \leq i-2$. Intuitively, these states are simply used to count the number of letters read so far. Hence, upon reaching q_j for any $j \leq i-1$, we know that we have read j letters. From here, the automaton has two transitions $q_{i-1} \xrightarrow{a} q_i^a$ and $q_{i-1} \xrightarrow{b} q_i^b$. Intuitively, these two transitions help us remember the i^{th} letter of the word.

Then, we have a collection of states $q_{i+1}^a, q_{i+2}^a ..., q_{2n-i}^a$ and $q_{i+1}^b, q_{i+2}^b, ..., q_{2n-i}^b$ along with the transitions, $q_j^a \xrightarrow{a,b} q_{j+1}^a$ and $q_j^b \xrightarrow{a,b} q_{j+1}^b$ for every $i \leq j \leq 2n-i-1$. Intuitively, these states are simply used to count the number of letters starting from the i^{th} letter, while simultaneously remembering the i^{th} letter. Hence, upon reaching q_j^a (resp. q_j^b) for any $j \leq 2n-i$, we know that we have read j letters and that the i^{th} letter that we read was an a (resp. a b). From here, we have two transitions $q_{2n-i}^a \xrightarrow{a} q_{2n-i+1}$ and $q_{2n-i}^b \xrightarrow{b} q_{2n-i+1}$. Intuitively, these two transitions force that the $(2n-i+1)^{th}$ letter that we read is the same as the i^{th} letter that we read before.

Finally, we have a sequence of states $q_{2n-i+1}, q_{2n-i+2} ..., q_{2n}$ with transitions $q_j \xrightarrow{a,b} q_{j+1}$ for every $2n-i+1 \leq j \leq 2n$. Once again, these states are simply used to count the number of letters read and we can show that if we reach q_j for any $j \leq 2n$, then we have read j letters. We then set the only final state to be q_{2n}.

(d) Suppose A is some NFA which recognizes L_n. For every $ww^R \in \Sigma^{2n}$, A has at least one accepting run on ww^R. Let q_w be the state reached by this run after reading the prefix w (If there are multiple such runs, pick any one of them). We claim that if $w \neq w'$, then $q_w \neq q_{w'}$. Notice that this claim implies that there are at least 2^n states in A and so it simply suffices to prove this claim.

Suppose $q_w = q_{w'}$ for some pair $w \neq w'$. Hence, after reading w' the automaton A can reach q_w. By definition of q_w, we know that there is a run on the word w^R starting from q_w and ending in a final state. This implies that the automaton accepts $w'w^R$, because first the automaton can reach q_w by reading w' and then from q_w it can reach a final state by reading w^R. But $w'w^R \notin L_n$, contradicting the fact that A recognizes L_n.

Puzzle exercise 2.4.

Let $\Sigma := \{a, b\}$. We say that a regular expression r is *ellellone*, if r is of the form st, s^*, x, $\boldsymbol{\epsilon}$, $\boldsymbol{\emptyset}$, $as + bt$, or $as + bt + \boldsymbol{\epsilon}$, where s, t are ellellone REs and $x \in \Sigma$.

For example, $(a+b)^*$ and $ab + b(ab + b\emptyset + \boldsymbol{\epsilon})^*$ are ellellone, but $(aa+ab)^*$ and $a + b(ab + (ba)^* + \boldsymbol{\epsilon})$ are not.

Prove that there is a regular language L, s.t. no ellellone RE r with $L(r) = L$ exists.

Hint: Consider the language L of words that contain an even number of a or an even number of b.