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Automata and Formal Languages

Winter Term 2023/24 – Exercise Sheet 1

Exercise 1.1.

Give a regular expression and an NFA for the language of all words over $\Sigma = \{a, b\}$ ...

(a) ... beginning and ending with a.

(b) ... such that the third letter from the right is a b.

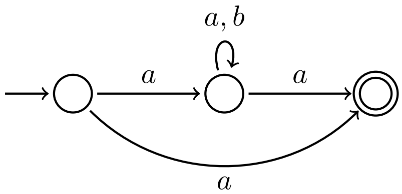
(c) ... that can be obtained from $babbab$ by deleting letters.

(d) ... with no occurrences of the subword bba.

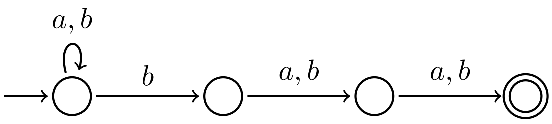
(e) ... with at most one occurrence of the subword bba.

Solution. We write Σ for $(a + b)$ and Σ^* for $(a + b)^*$.

(a) $a + (a\Sigma^*a)$

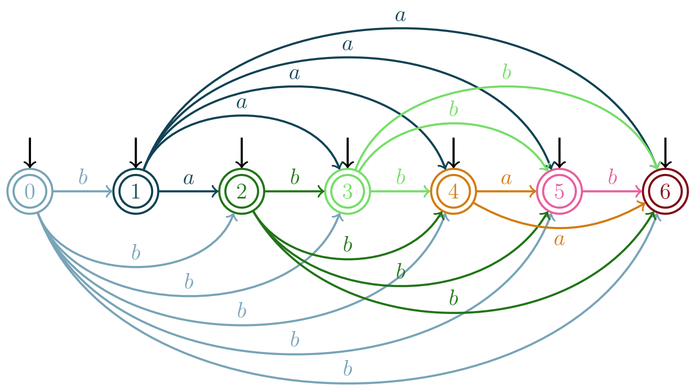


(b) $\Sigma^*b\Sigma\Sigma$

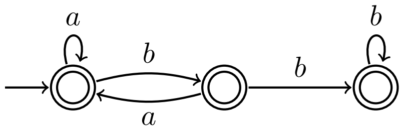


(c) $(b + \epsilon)(a + \epsilon)(b + \epsilon)(b + \epsilon)(a + \epsilon)(b + \epsilon)$

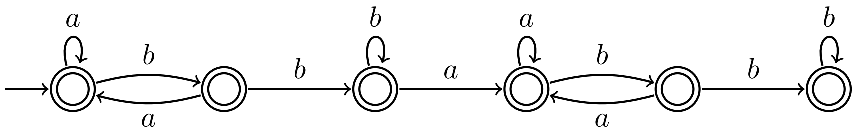
One possible NFA for the language is the following. Note that every state of this NFA is initial and accepting. There are 7 states, labelled by 0, 1, 2, 3, 4, 5 and 6. From 0, upon reading a b, we can go to any state strictly bigger than 0; From 1, upon reading an a, we can go to any state strictly bigger than 1, and so on.



(d) $(a + ba)^* b^*$



(e) $((a + ba)^* b^*) + ((a + ba)^* b^* (bba)(a + ba)^* b^*)$



Exercise 1.2.

Let A, B and C be three languages.

(a) Prove that if $A \subseteq BC$ then $A^* \subseteq (B^* + C^*)^*$. Is the converse true?

(b) Prove that the languages of $((a + ba)^* + b^*)^*$ and $(a + b)^*$ are the same.

Solution.

(a) Suppose $A \subseteq BC$. First, we show that $A^* \subseteq (BC)^*$. Indeed, if $w \in A^*$, then w can be decomposed as $w_1 w_2 ... w_n$ for some number n such that each $w_i \in A$. Since $A \subseteq BC$, it follows that each $w_i \in BC$ and so $w \in (BC)^*$.

Now, we show that $(BC)^* \subseteq (B^* + C^*)^*$. If $w \in (BC)^*$ then w can be decomposed as $w_1 w_2 ... w_n$ for some number n such that each $w_i \in BC$. Since each $w_i \in BC$, it follows that each w_i can be further decomposed as $u_i v_i$ where $u_i \in B$ and $v_i \in C$. Hence $w = u_1 v_1 u_2 v_2 ... u_n v_n$ and since each $u_i, v_i \in B + C \subseteq B^* + C^*$, it follows that $w \in (B^* + C^*)^*$.

(b) Let $U = (a + b), V = (a + ba)^*$ and $W = b^*$. We then have that $U \subseteq VW$ and so by the previous subpart, we have that $U^* \subseteq (V^* + W^*)^*$. Since $V^* = V$ and $W^* = W$, it follows that $(a + b)^* \subseteq ((a + ba)^* + b^*)^*$. Further, since $(a + b)^*$ is the set of all words over $\{a, b\}$, we have that $((a + ba)^* + b^*)^* \subseteq (a + b)^*$. The desired claim then follows.

Exercise 1.3.

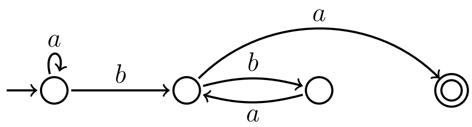
Consider the language $L \subseteq \{a, b\}^*$ given by the regular expression $a^*b(ba)^*a$.

(a) Give an NFA that accepts L.

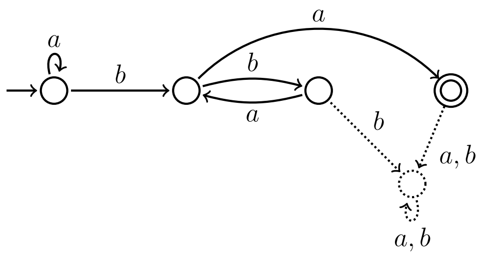
(b) Give a DFA that accepts L.

Solution.

(a) NFA accepting L



(b) DFA accepting L



Exercise 1.4.

Let $\Sigma = \{a, b\}$ and let $\Sigma^* = (a + b)^*$. Suppose $w = a_1a_2...a_n$ where each $a_i \in \Sigma$. Then the *upward closure* of a word w is defined as the set

$$\uparrow w = \{u_1a_1u_2a_2...u_na_nu_{n+1} : u_1, u_2, ..., u_{n+1} \in \Sigma^*\}$$

The *upward closure* of a language L is defined as the set $\uparrow L = \cup_{w \in L} \uparrow w$.

(a) Give an algorithm that takes as input a regular expression r and outputs a regular expression $\uparrow r$ such that $\mathcal{L}(\uparrow r) = \uparrow(\mathcal{L}(r))$.

(b) Give an algorithm that takes as input an NFA A and outputs an NFA B with exactly the same number of states as A such that $\mathcal{L}(B) = \uparrow\mathcal{L}(A)$.

Solution.

(a) We define $\uparrow r$ by induction on the regular expression r:

- If $r = \emptyset$, then we set $\uparrow r = \emptyset$
- If $r = \epsilon$, then we set $\uparrow r = \Sigma^*$
- If $r = x$ for some $x \in \{a, b\}$, then we set $\uparrow r = \Sigma^* x \Sigma^*$
- If $r = r_1 + r_2$ for some r_1 and r_2, then we set $\uparrow r = (\uparrow r_1) + (\uparrow r_2)$
- If $r = r_1 r_2$ for some r_1 and r_2, then we set $\uparrow r = (\uparrow r_1)(\uparrow r_2)$
- If $r = (r_1)^*$ for some r_1, then we set $\uparrow r = \Sigma^*$. Note that if $r = (r_1)^*$ for some r_1, then $\epsilon \in \mathcal{L}(r)$ and so $\uparrow\mathcal{L}(r)$ must contain every word.

(b) Let A be an NFA recognizing a language L. We construct the NFA B from A as follows: Corresponding to every state q of A and every letter $x \in \{a, b\}$, we add a self-loop transition (q, x, q). These new transitions will be referred to as *special transitions*. We now claim that $\mathcal{L}(B) = \uparrow L$.

Suppose $w \in \uparrow L$. Hence, $w = u_1 a_1 u_2 a_2 ... u_n a_n u_{n+1}$ for some words $u_1, ..., u_{n+1}$ and letters $a_1, ..., a_n$ such that $w' := a_1 a_2 ... a_n \in L$. Hence, there is an accepting run $\rho := q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 ... q_{n-1} \xrightarrow{a_n} q_n$ of A on the word w'. Now, notice that $q_0 \xrightarrow{u_1} q_0 \xrightarrow{a_1} q_1 \xrightarrow{u_2} q_1 \xrightarrow{a_2} q_1 ... q_{n-1} \xrightarrow{a_n} q_n \xrightarrow{u_{n+1}} q_n$ is an accepting run of B on the word w. (Here $q_i \xrightarrow{u_{i+1}} q_i$ denotes that starting from the state q_i, there is a run on the word u_{i+1} which ends at q_i). This implies that $w \in \mathcal{L}(B)$.

Suppose ρ is an accepting run of B on the word w. We now prove that $w \in \uparrow L$ by induction on the number of special transitions of ρ. If ρ has no special transitions, then ρ is also a run of A on w and so $w \in L \subseteq \uparrow L$. For the induction step, suppose ρ has $k + 1$ special transitions for some $k \geq 0$. Let $w = w_1 w_2 ... w_n$ with each $w_i \in \Sigma$ and let $\rho = q_0 \xrightarrow{w_0} q_1 \xrightarrow{w_1} q_2 ... q_{n-1} \xrightarrow{w_n} q_n$. Let $q_i \xrightarrow{w_{i+1}} q_{i+1}$ be the first special transition along ρ. Since this is a special transition, it must be the case that $q_i = q_{i+1}$. Let w' be the word obtained from w by deleting the letter w_{i+1} at the $(i+1)^{th}$ position and let ρ' be the accepting run of B on w' obtained from ρ by deleting the transition $q_i \xrightarrow{w_{i+1}} q_i$. Since ρ' has only k special transitions, by induction hypothesis, $w' \in \uparrow L$. Since w can be obtained from w' by adding a letter, it follows that $w \in \uparrow L$ as well, thereby finishing the proof.