Petri nets — Exercise sheet 8

Solution published on 21.07.2020

Exercise 8.1
Recall that a Petri net N is *well-formed* if there is a marking M_0 such that (N, M_0) is live and bounded. Consider the following proposition (full proof given in Proposition 5.4 of "Free Choice Petri Nets" by J. Desel and J. Esparza):

Proposition 1. *Let N be a well-formed free-choice net and R be a minimial siphon of N. Then*

(1) R is a trap of N.

(2) The subnet generated by $(R, {}^\bullet R)$ is an S-component of N.

Use the above proposition, as well as Commoner's Liveness Theorem and Hack's Boundedness Theorem, to prove or disprove the following:

1. A bounded free-choice system (N, M_0) is live iff every minimal siphon of N is a trap marked at M_0.
2. A live free-choice system (N, M_0) is bounded iff every minimal siphon of N is a trap marked at M_0.

Exercise 8.2
Let $(\mathcal{N}, M_0)$ be a bounded and strongly connected free-choice system which is deadlock-free, where $\mathcal{N} = (P, T, F)$. For every $M \in \mathbb{N}^P$, let $d(M)$ be the number of transitions dead at M. Let $K \in \mathbb{N}^P$ be such that $d(K) = \max\{d(M) : M_0 \xrightarrow{*} M\}$.

(a) Let $u \in T$ be a transition not dead at K. Show that there exists an infinite firing sequence $\sigma \in T^\omega$ enabled at K and containing infinitely many occurrences of u.
Hint: Use the fact that $d(K)$ is maximal.

(b) Let $u, v \in T$ be such that u is not dead at K and $v \in (u^\bullet)^\bullet$. Show that v is not dead at K.
Hint: Use (a).

(c) Show that there exists a path $\gamma \in (T \cup P)^*$ of $\mathcal{N}$ such that γ contains all transitions of T and γ starts with a transition enabled at K.

(d) Use (b) and (c) to show that $d(K) = 0$, and hence that $(\mathcal{N}, M_0)$ is live.

Exercise 8.3
Recall the rank theorem from the lecture:

Theorem 5.3.16. *A free-choice system (N, M_0) is live and bounded iff*

1. *N has a positive S-invariant.*
2. *N has a positive T-invariant.*

3. *The rank of the incidence matrix* $\boldsymbol{N}$ *is equal to* $c-1$*, where* c *is the number of clusters of* N*.*
4. *Every proper siphon of* N *is marked under* M_0*.*

We want to show that all hypotheses of the theorem are necessary, by giving four counter-examples where each hypothesis save one holds on each counter-example.

More precisely, give four free-choice systems $(N_1, M_1), (N_2, M_2), (N_3, M_3), (N_4, M_4)$ such that

(a) conditions 2,3,4 hold on (N_1, M_1), but N_1 has no positive S-invariant;

(b) conditions 1,3,4 hold on (N_2, M_2), but N_2 has no positive T-invariant;

(c) conditions 1,2,4 hold on (N_3, M_3), but the rank of $\boldsymbol{N_3}$ is not equal to the number of clusters of N_3; and

(d) conditions 1,2,3 hold on (N_4, M_4), but some siphon of N_4 is not marked under M_4.

Note that none of these systems can be both live and bounded. Which property is violated on each system?

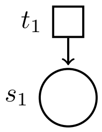
Solution 8.1

1. (⇒) Let (N, M_0) be a live and bounded free-choice system. Then N is well-formed, and by Proposition **??**, every minimal siphon of N is a trap. By Commoner's Liveness Theorem, every minimal siphon contains a trap marked at M_0, therefore every minimal siphon of N is also a trap marked at M_0.

   (⇐) Let (N, M_0) be a bounded free-choice system where every minimal siphon of N is a trap marked at M_0. Then every minimal siphon contains a trap marked at M_0, and by Commoner's Liveness Theorem, the system is live.

2. The (⇒) direction holds as in 1, however the other direction does not. Even though we can infer with Proposition 6.3.1 that every minimal siphon generates an S-component of N, we can not show that every place belongs to a minimal siphon and therefore to an S-component, which would be necessary for Hack's Boundedness Theorem.

   The following live free-choice system is a counterexample for this conjecture. It has no minimal siphons, therefore every minimal siphon is a trap marked at M_0, however it is unbounded.



Solution 8.2

(a) Since u is not dead at K, there exist $\sigma_1 \in T^*$ and $L_1 \in \mathbb{N}^P$ such that $K \xrightarrow{\sigma_1} L_1$ and u is enabled at L_1. Let $L_1' \in \mathbb{N}^P$ be such that $L_1 \xrightarrow{u} L_1'$. By maximality of $d(K)$, we have $d(L_1') \leq d(K)$. Moreover, every transition dead at K is also dead at L_1', and hence the transitions dead at K and L_1' are the same. Therefore, u is not dead at L_1'. This implies that there exist $\sigma_2 \in T^*$ and $L_2 \in \mathbb{N}^P$ such that $L_1' \xrightarrow{\sigma_2} L_2$ and u is enabled at L_2. By repeating this argument, we obtain an infinite firing sequence $\sigma = \sigma_1 u \sigma_2 u \cdots$ enabled at K and containing infinitely many occurrences of u. □

(b) By (a), there exists an infinite firing sequence $\sigma \in T^\omega$ which is enabled at K and such that σ contains infinitely many occurrences of u. Since $v \in (u^\bullet)^\bullet$, there exists $p \in P$ such that $(u, p), (p, v) \in F$. Since $(\mathcal{N}, M_0)$ is bounded and u produces a token in p, σ must contain infinitely many occurrences of a transition w such that $p \in {}^\bullet w$. Indeed, otherwise p would be unbounded. In particular, this implies that w is not dead at K. Since $\mathcal{N}$ is free-choice, v is enabled at the same markings as w. Therefore, v is not dead at K. □

(c) Since $(\mathcal{N}, M_0)$ is deadlock-free and K is reachable from M_0, there exists a transition t enabled at K. Moreover, since $\mathcal{N}$ is strongly connected, there exists a path starting in t that goes through all nodes of $\mathcal{N}$. □

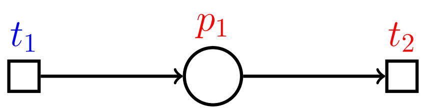
(d) By (c), there exists a path γ that contains all transitions of $\mathcal{N}$ and whose first transition is enabled at K. Let $\gamma = t_1 p_1 t_2 \cdots p_{n-1} t_n$. We claim that, for every $1 \leq i \leq n$, t_i is not dead at K. It follows from this claim that $d(K) = 0$, and hence, by maximality of $d(K)$, that $(\mathcal{N}, M_0)$ is live.

   We prove the claim by induction on i. The base case follows immediately. Let $i > 1$ and assume that the claim holds for t_{i-1}. We have $(t_{i-1}, p_{i-1}), (p_{i-1}, t_i) \in F$. Therefore, $t_i \in (t_{i-1}^\bullet)^\bullet$. By induction hypothesis, t_{i-1} is not dead at K and hence, by (b), t_i is also not dead. □

Solution 8.3

In the following free-choice nets, the colours of the nodes indicate their cluster.

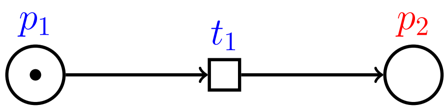
(a) Let (N_1, M_1) be



incidence matrix	positive S-inv	positive T-inv	rank=c-1	siphons
$\begin{pmatrix}1 & -1\end{pmatrix}$		$\begin{pmatrix}1 & 1\end{pmatrix}$	$1 = 2 - 1$	$\emptyset$

If there were a positive S-invariant, then by Proposition 4.3.8 the system would be bounded. But this is not the case. The system (N_1, M_1) is live but unbounded.

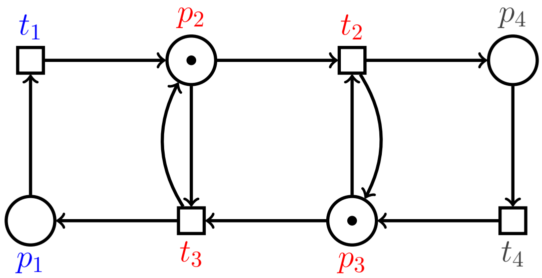
(b) Let (N_2, N_2) be



incidence matrix	positive S-inv	positive T-inv	rank=c-1	siphons
$\begin{pmatrix}-1 \\ 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\end{pmatrix}$		$1 = 2 - 1$	$\{p_1\}, \{p_1, p_2\}$

Since ${}^\bullet p_1 = \emptyset$ and $p_1^\bullet = \{t_1\}$, all T-invariants J must be such that $J(t_1) = 0$ and so cannot be positive. The system (N_2, M_2) is bounded but not live.

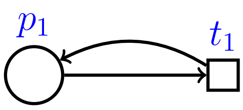
(c) Let (N_3, M_3) be



incidence matrix	positive S-inv	positive T-inv	rank=c-1	siphons
$\begin{pmatrix}-1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}$	$3 \neq 3 - 1$	$\{p_3, p_4\}, \{p_1, p_2\}$

The system (N_3, M_3) is bounded but not live (we reach a deadlock by firing either $t_2 t_4$ or $t_3 t_1$).

(d) Let (N_4, M_4) be



incidence matrix	positive S-inv	positive T-inv	rank=c-1	siphons
$\begin{pmatrix}0\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$0 = 1 - 1$	$\{p_1\}$ is not marked!

The system (N_4, M_4) is bounded but not live.