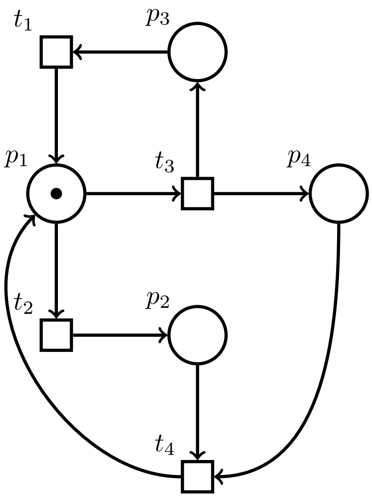
Petri nets — Exercise sheet 4

Solution to be published on 09.06.2020

Exercise 4.1

Consider the following net $\mathcal{N} = (P, T, F)$:



(a) Draw a coverability graph for $(\mathcal{N}, \{p_1\})$.

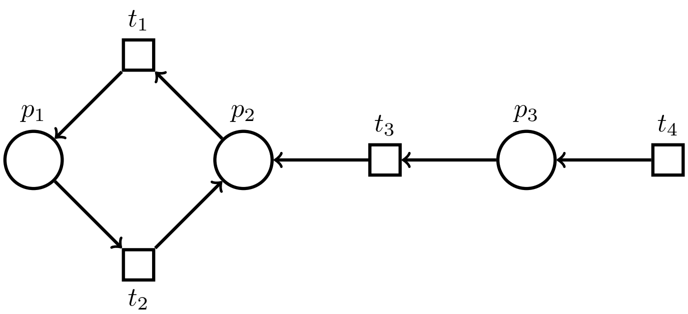
(b) Is $(\mathcal{N}, \{p_1\})$ bounded? If so, why? If not, which places are bounded?

(c) Describe the set of markings coverable from $\{p_1\}$.

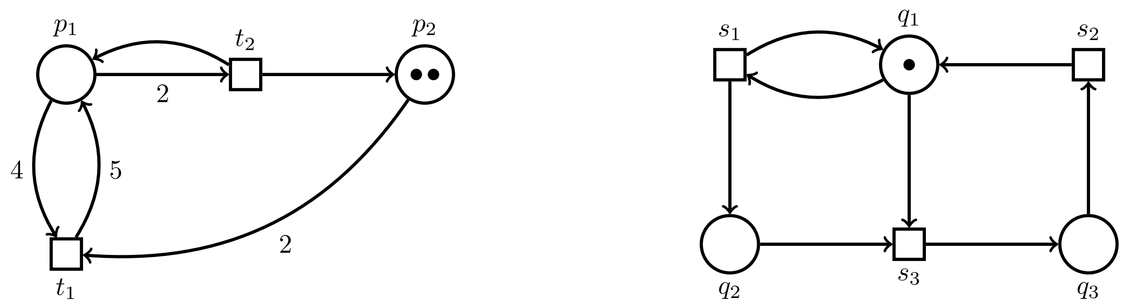
(d) We say that a Petri net $(\mathcal{N}, M_0)$ terminates if all its firing sequences are finite. Does $(\mathcal{N}, \{p_1\})$ terminate? Justify your answer.

Exercise 4.2

The algorithm Coverability-Graph does not specify how the coverability graph should be traversed during its construction. Show that different traversal strategies can lead to different coverability graphs. More precisely, exhibit a marking M and two different coverability graphs for $(\mathcal{N}, M)$, where $\mathcal{N}$ is the following net:



Exercise 4.3
Let $\mathcal{N}$ and $\mathcal{N}'$ be respectively the left and right Petri nets below.



Use the backward algorithm to answer the following questions.

(a) Describe the set of initial markings from which $M = \{2 \cdot p_2\}$ is coverable in $\mathcal{N}$. Record all intermediate sets of markings with their finite representation of minimal elements.

(b) Determine whether $\{q_1, q_3\}$ is coverable from $\{q_1\}$ in $\mathcal{N}'$.

(c) Determine whether $\{q_1, q_2\}$ is coverable from $\{q_1\}$ in $\mathcal{N}'$.

Exercise 4.4
Give a procedure to decide the following problem:

Given a Petri net $(\mathcal{N}, M_0)$ and a transition sequence σ, is there a transition sequence σ' such that $\sigma'\sigma$ is enabled at M_0?

For the procedure, you may use any already known decision procedures and algorithms such as coverability graph and backwards reachability, or you may adapt those algorithms or use parts of them.

Exercise 4.5
A net with reset, doubling and transfer arcs is a tuple (P, T, F, R, D, Tr) where (P, T, F) is a net,

$$R \subseteq P \times T,\ D \subseteq T \times P,\ Tr \subseteq (P \times T) \cup (T \times P),$$

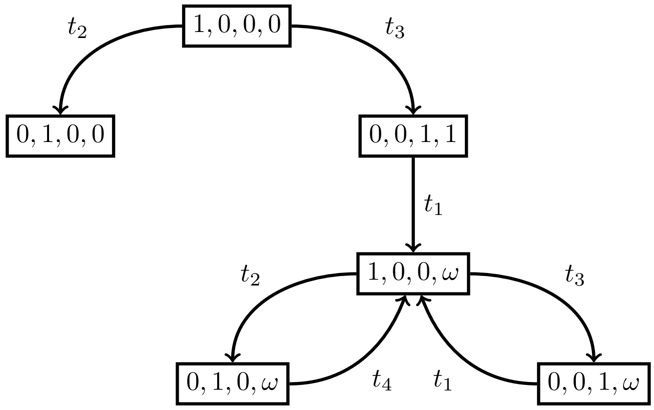
and F, R, D and Tr are pairwise disjoint. Let $M \in \mathbb{N}^P$ and $t \in T$. We say that t is enabled at M if $M(p) > 0$ for every $(p, t) \in F$. Firing t at M has the following effect:

- every arc $(p, t) \in F$ consumes a token from p;
- every arc $(t, p) \in F$ produces a token in p;
- every arc $(p, t) \in R$ empties p;
- every arc $(t, p) \in D$ doubles the amount of tokens in p;
- every arc $(p, t) \in Tr$ empties p;
- every arc $(t, p) \in Tr$ adds $\sum_{(q,t)\in Tr} M(q)$ tokens to p.

Show that the backward algorithm works for this extended class of nets by showing that it is monotonic, i.e. show that for every markings $X, X', Y \in \mathbb{N}^P$, if $X \to Y$ and $X' \geq X$, then $X' \to Y'$ for some $Y' \geq Y$.

Solution 4.1

(a) The following is a coverability graph where nodes are labeled with respect to the total order $p_1 < p_2 < p_3 < p_4$:



(b) It is not bounded since some markings of the graph contain ω. Places p_1, p_2 and p_3 are bounded because no marking of the graph contains an ω in the three first components.

★ This can also be tested with `LoLA` as follows:

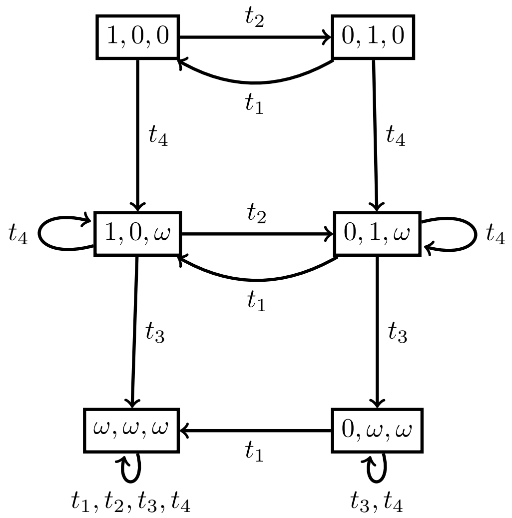
```
> lola pn_2-4.lola -f "AG (p1 < oo)" --search=cover
lola:  result:  yes
lola:  The net satisfies the given formula.


> lola pn_2-4.lola -f "AG (p2 < oo)" --search=cover
lola:  result:  yes
lola:  The net satisfies the given formula.


> lola pn_2-4.lola -f "AG (p3 < oo)" --search=cover
lola:  result:  yes
lola:  The net satisfies the given formula.


> lola pn_2-4.lola -f "AG (p4 < oo)" --search=cover
lola:  result:  no
lola:  The net does not satisfy the given formula.
```

(c) $\{M \in \mathbb{N}^P : M(p_1) + M(p_2) + M(p_3) = 1\}$.

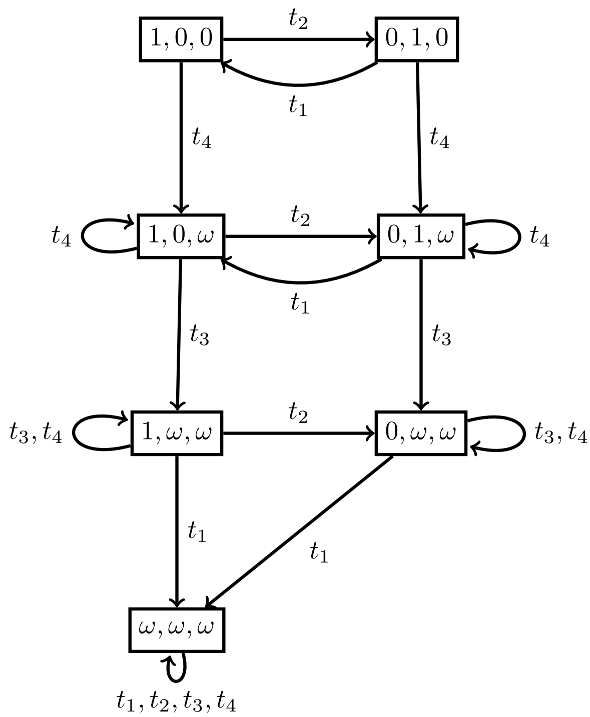
(d) No, it does not terminate. A Petri net does not terminate if and only if its coverability graph contains a cycle or at least one node with an ω. The above coverability graph contains both. In particular, $t_3 t_1 (t_2 t_4)^\omega$ is an infinite firing sequence.

Solution 4.2

Let $M = \{p_1\}$. We exhibit two coverability graphs for $(\mathcal{N}, M)$, where nodes are labeled with respect to the total order $p_1 < p_2 < p_3$. We construct the first coverability graph by exploring the configuration $(1, 0, 0)$ and its enabled transitions, then configuration $(0, 1, 0)$ and then $(1, 0, \omega)$:



For the second coverability graph, we explore the configuration $(1, 0, 0)$ and its enabled transitions, then configuration $(1, 0, \omega)$ and then $(0, 1, 0)$:



Treating configuration $(1, 0, \omega)$ and its enabled transitions before $(0, 1, 0)$ allows configuration $(1, \omega, \omega)$ to be added to the graph.

Solution 4.3

(a) We execute the backward algorithm from $M = (0, 2)$. In order to build the whole set of initial markings, we ignore the stopping criterion based on M_0.

For the backwards reachability algorithm, it can be helpful to construct the reverse net, with all arcs inverted, to compute the predecessors of markings. Then start with the target marking M and add tokens as necessary for firing transitions.

We start with $m_0 = \{(0, 2)\}$, which is the set of minimal elements for $\{M\}$. Recall that the backwards reachability algorithm iteratively updates m to

$$m = min(m \cup \bigcup_{t \in T} pre(R[t] \wedge m, t)).$$

We have $R[t_1] = (5, 0)$ and $R[t_2] = (1, 1)$.

$$\begin{aligned} pre(R[t_1] \wedge (0, 2), t_1) &= pre((5, 2), t_1) = (4, 4) \\ pre(R[t_2] \wedge (0, 2), t_2) &= pre((1, 2), t_2) = (2, 1) \end{aligned}$$

After adding the new markings to m_0 and eliminating non-minimal markings, our new set is $m_1 = \{(0, 2), (2, 1)\}$. For the new marking $(2, 1)$, we compute the predecessors:

$$\begin{aligned} pre(R[t_1] \wedge (2, 1), t_1) &= pre((5, 1), t_1) = (4, 3) \\ pre(R[t_2] \wedge (2, 1), t_2) &= pre((2, 1), t_2) = (3, 0) \end{aligned}$$

We add the new markings, take the minimal elements and obtain $m_2 = \{(0, 2), (2, 1), (3, 0)\}$. For $(3, 0)$, we compute the predecessors:

$$\begin{aligned} pre(R[t_1] \wedge (3, 0), t_1) &= pre((5, 0), t_1) = (4, 2) \\ pre(R[t_2] \wedge (3, 0), t_2) &= pre((3, 1), t_2) = (4, 0) \end{aligned}$$

No new minimal markings are obtained, so we have reached a fixpoint. The set of initial markings is $\{M \in \mathbb{N}^2 : M \geq (0, 2) \text{ or } M \geq (2, 1) \text{ or } M \geq (3, 0)\}$.

(b) We want to determine whether $M = (1, 0, 1)$ is coverable from $M_0 = (1, 0, 0)$. It is not the case, since executing the backward algorithm from M does not generate any marking less or equal to M_0. We note $\text{pre}_{s_i}(m_i)$ for $pre(R[s_i] \wedge m_i, s_i)$, where s_i a transition and m_i a marking.

Iteration	pre(m)	m
0	—	$\{(1, 0, 1)\}$
1	$\text{pre}_{s_1}(1, 0, 1) = (1, 0, 1)$ $\text{pre}_{s_2}(1, 0, 1) = (0, 0, 2)$ $\text{pre}_{s_3}(1, 0, 1) = (2, 1, 0)$	$\{(1, 0, 1), (0, 0, 2), (2, 1, 0)\}$
2	$\text{pre}_{s_1}(1, 0, 1) = (1, 0, 1)$ $\text{pre}_{s_2}(1, 0, 1) = (0, 0, 2)$ $\text{pre}_{s_3}(1, 0, 1) = (2, 1, 0)$ $\text{pre}_{s_1}(0, 0, 2) = (1, 0, 2)$ $\text{pre}_{s_2}(0, 0, 2) = (0, 0, 3)$ $\text{pre}_{s_3}(0, 0, 2) = (1, 1, 1)$ $\text{pre}_{s_1}(2, 1, 0) = (2, 0, 0)$ $\text{pre}_{s_2}(2, 1, 0) = (1, 1, 1)$ $\text{pre}_{s_3}(2, 1, 0) = (3, 2, 0)$	$\{(1, 0, 1), (0, 0, 2), (2, 0, 0)\}$
3	$\text{pre}_{s_1}(2, 0, 0) = (2, 0, 0)$ $\text{pre}_{s_2}(2, 0, 0) = (1, 0, 1)$ $\text{pre}_{s_3}(2, 0, 0) = (3, 1, 0)$	$\underbrace{\{(1, 0, 1), (0, 0, 2), (2, 0, 0)\}}_{\text{unchanged}}$

(c) We want to determine whether $M' = (1,1,0)$ is coverable from $M_0 = (1,0,0)$. It is the case, since executing the backward algorithm from M' yields M_0 after one iteration:

Iteration	pre(m)	m
0	—	$\{(1,1,0)\}$
1	$\text{pre}_{s_1}(1,1,0) = (1,0,0)$ $\text{pre}_{s_2}(1,1,0) = (0,1,1)$ $\text{pre}_{s_3}(1,1,0) = (2,2,0)$	$\{\underbrace{\mathbf{(1,0,0)}}_{\leq M_0}, (0,1,1)\}$

Solution 4.4

Let $\sigma = t_1 t_2 \dots t_n$. Consider the following algorithm:

Algorithm 0.1: Suffix Existence

$m \leftarrow \{\mathbf{0}\}$
for i from n to 1 **do**
  $m \leftarrow \min(pre(m, t_i))$
if $\exists M \in m : M$ is coverable from M_0 **then**
  return there is a sequence σ' s.t $M_0 \xrightarrow{\sigma'\sigma}$
else
  return there is no sequence σ' s.t $M_0 \xrightarrow{\sigma'\sigma}$

Basically, we fire σ backwards, starting from all markings (the upward closed set with the zero marking as the minimal element). We obtain the set m of all markings where σ is enabled (again an upward closed set). We then check if one the minimal markings of the set m is coverable from M_0, and if yes, there is a sequence σ' from which we can reach an element of m, and thus fire σ.

The existance of a coverable element in m can be checked with the coverability graph or the backwards reachability algorithm. With the backwards reachability algorithm, we can directly start with m as the initial set of minimal markings.

Solution 4.5

Let $X, X', Y \in \mathbb{N}^P$ and $t \in T$ be such that $X \xrightarrow{t} Y$ and $X' \geq X$. Let us first argue that t is enabled at X':

$$
\begin{aligned}
t \text{ is enabled at } X &\iff X(p) > 0 \text{ for every } (p,t) \in F \\
&\implies X'(p) > 0 \text{ for every } (p,t) \in F \qquad\qquad (\text{since } X' \geq X) \\
&\iff t \text{ is enabled at } X'.
\end{aligned}
$$

Let $Y' \in \mathbb{N}^P$ be the marking such that $X' \xrightarrow{t} Y'$. Let $p \in P$. It remains to show that $Y'(p) \geq Y(p)$. We only prove the case where p is not in both the preset and postset of t:

Case	Proof
$(p,t) \in F$	$Y'(p) = X'(p) - 1 \geq X(p) - 1 = Y(p)$
$(t,p) \in F$	$Y'(p) = X'(p) + 1 \geq X(p) + 1 = Y(p)$
$(p,t) \in R$	$Y'(p) = 0 = Y(p)$
$(t,p) \in D$	$Y'(p) = 2 \cdot X'(p) \geq 2 \cdot X'(p) = Y(p)$
$(p,t) \in Tr$	$Y'(p) = 0 = Y(p)$
$(t,p) \in Tr$	$Y'(p) = X'(p) + \sum_{(q,t) \in Tr} X'(q) \geq X(p) + \sum_{(q,t) \in Tr} X(q) = Y(p)$