Petri nets — Exercise sheet 3

Solution to be published on 26.05.2020

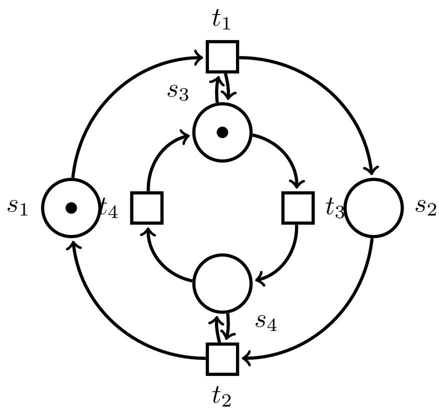
The symbol ★ denotes that a question in an exercise is harder than usual.

Exercise 3.1

(a) Give a net $\mathcal{N}$ and two markings M and M' such that $M \leq M'$, $(\mathcal{N}, M)$ is bounded, and $(\mathcal{N}, M')$ is *not* bounded.

(b) Give a net $\mathcal{N}$ and two markings M and M' such that $M \leq M'$, $(\mathcal{N}, M)$ is deadlock-free, and $(\mathcal{N}, M')$ is *not* deadlock-free.

(c) Give a net $\mathcal{N}$ and two markings M and M' such that $M \leq M'$, $(\mathcal{N}, M)$ is bounded *and* live, and $(\mathcal{N}, M')$ is *not* bounded. *Hint:* Add a place and arcs to the following net to obtain a solution:



Exercise 3.2

Let $\mathcal{N} = (P, T, W)$ be a net with weighted arcs. Let $M, M' \in \mathbb{N}^P$, $\sigma, \sigma' \in T^*$ and $t \in T$ be such that $M \xrightarrow{\sigma t \sigma'} M'$. Prove or disprove the following statements:

(a) if t does not consume any token, i.e $W(p,t) = 0$ for every $p \in P$, then $M \xrightarrow{t \sigma \sigma'} M'$.

(b) if t does not produce any token, i.e. $W(t,p) = 0$ for every $p \in P$, then $M \xrightarrow{\sigma \sigma' t} M'$.

(c) if t consumes no more tokens than it produces, i.e $W(p,t) \leq W(t,p)$ for every $p \in P$, then $M \xrightarrow{t \sigma \sigma'} M'$.

(d) if t produces no more tokens than it consumes, i.e. $W(t,p) \leq W(p,t)$ for every $p \in P$, then $M \xrightarrow{\sigma \sigma' t} M'$.

Exercise 3.3

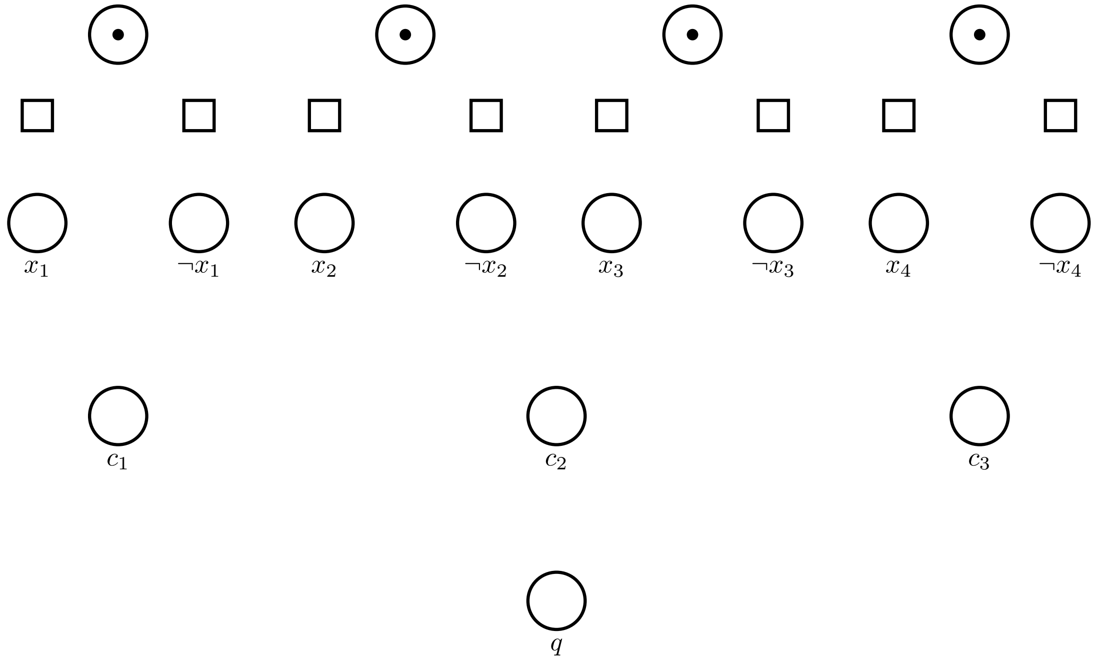
(a) You have seen that the coverability problem is PSPACE-hard, even for 1-bounded Petri nets (Section 3.1.1 of the Script). Show that the following problems are also PSPACE-hard by reducing 1-bounded Petri net coverability to them (you do not need to prove the correctness of your reduction). Given a 1-bounded Petri net:

1. is there a reachable marking that concurrently enables two given transitions t_1 and t_2?
2. can a given transition t ever occur?
3. is there a run containing a given transition t infinitely often?

(b) Show that problems 1. and 2. can be reduced to coverability in a 1-bounded Petri net. This will prove that they are solvable in PSPACE, since coverability is PSPACE-complete in 1-bounded Petri nets.

Exercise 3.4

(a) Recall that 3-SAT is the problem of determining the satisfiabillity of a Boolean formula, in conjunctive normal form, whose clauses have at most three literals. It is well-known that 3-SAT is NP-complete. Give a polynomial time reduction from 3-SAT to Petri net coverability. You can simply illustrate your reduction for the formula $\varphi(x_1, x_2, x_3, x_4) = (x_1 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_3 \vee \neg x_4)$ by extending the following partial Petri net in such a way that φ is satisfiable if and only if $\{q\}$ is coverable:



(b) Adapt your previous reduction to non-boundedness instead of coverability.

(c) ★ Give a polynomial time reduction from coverability to reachability.
Hint: the markings that show coverability can be left unchanged, and the Petri net can be transformed by adding new transitions *only*.

(d) ★ Prove that the reduction you gave in (c) is correct.
Hint: make use of #3.2.

Exercise 3.5

We define *labeled nets* as $\mathcal{N} = (S, T, F, \Sigma, s_{\mathcal{N}})$, where (S, T, F) is a classic net, Σ is a finite alphabet of *labels* and $s_{\mathcal{N}} : T \to \Sigma$ is called the *signature* of $\mathcal{N}$. Notice that the signature can assign the same label to two different transitions. The signature can be extended to a function $s_{\mathcal{N}}$ from T^* to Σ^*, using $s_{\mathcal{N}}(\sigma t) = s_{\mathcal{N}}(\sigma) s_{\mathcal{N}}(t)$ for $t \in T, \sigma \in T^*$. We call *free* a labeled net such that $\Sigma = T$ and $s_{\mathcal{N}}$ is the identity.

For $M_0 \in \mathbb{N}^S$ a marking, we call $(\mathcal{N}, M_0)$ a *labeled Petri net.* For $\mathcal{C}_t$ a set of markings, we call $(\mathcal{N}, M_0, \mathcal{C}_t)$ a *terminal labeled Petri net*, and $\mathcal{C}_t$ is its set of *terminal markings.* We define

$$L_t(\mathcal{N}) = \{w \in \Sigma^* | \exists \sigma \in T^*, \exists C \in \mathcal{C}_t . M_0 \xrightarrow{\sigma} C \wedge s_{\mathcal{N}}(\sigma) = w\}$$

the *terminal language* for $(\mathcal{N}, M_0, \mathcal{C}_t)$.

(a) Show that regular languages (defined by automata) are included in terminal languages. *Hint:* Think of how to transform a DFA into a labeled net.

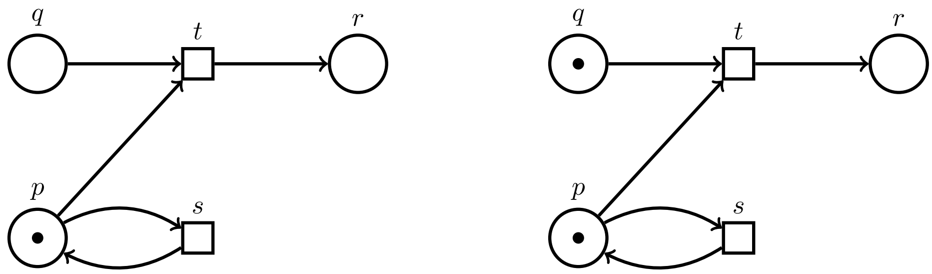
(b) Terminal languages can express more than just regular languages. For instance, show that the non context-free language $\{a^n b^n c^n | n \in \mathbb{N}, n \geq 1\}$ is a terminal language for a certain terminal labeled Petri net with a finite terminal set of markings.

(c) If we consider only *free* terminal labeled Petri nets, that is terminal labeled Petri nets without duplication of labels, then there are regular languages that are not terminal languages. There are even finite languages that are not terminal languages, for example $L = \{abc, ba\}$. Show that L is not a terminal language for any free terminal labeled Petri net.

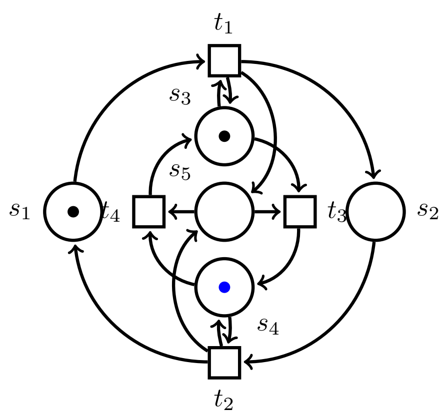
Solution 3.1

(a) The following net is bounded from the empty marking since its reachability set is empty. However, it is not bounded from $\{p\}$ since repetitively firing t increases the number of tokens in q.

(b) The following net is deadlock-free from $\{p\}$ since s is always enabled. However, it is not deadlock-free from $\{p, q\}$ since $\{p, q\} \xrightarrow{t} \{r\}$ and $\{r\}$ is dead.



(c) The following Petri net is live and bounded with the black tokens, but not bounded with the additional blue token in s_4, as repeatedly firing $t_1 t_2$ can put an arbitrary number of tokens in s_5.



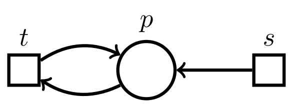
Solution 3.2

(a) True. Let $A, A' \in \mathbb{N}^P$ be such that $M \xrightarrow{\sigma} A \xrightarrow{t} A' \xrightarrow{\sigma'} M'$. Since $W(p, t) = 0$ for every $p \in P$, t is enabled at any marking. We have $A' - A \geq \mathbf{0}$ with $(A' - A)(p) = W(t, p)$ for every $p \in P$. Thus, $M \xrightarrow{t} M + (A' - A)$ and, by monotonicity, $M + (A' - A) \xrightarrow{\sigma} A + (A' - A)$. Therefore,

$$M \xrightarrow{t} M + (A' - A) \xrightarrow{\sigma} A' \xrightarrow{\sigma'} M'.$$

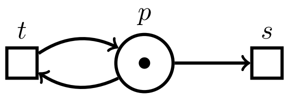
(b) True. Symmetrically as above.

(c) False. Consider the following Petri net:



We have $0 \xrightarrow{st} 1$ and $W(p, t) = W(t, p)$, yet ts cannot be fired from 0.

(d) False. Consider the following Petri net:



We have $1 \xrightarrow{ts} 0$ and $W(t,p) = W(p,t)$, yet st cannot be fired from 1.

Solution 3.3

(a) Let $(\mathcal{N}, M_0)$ be a 1-bounded Petri net, and let M be the marking we wish to cover.

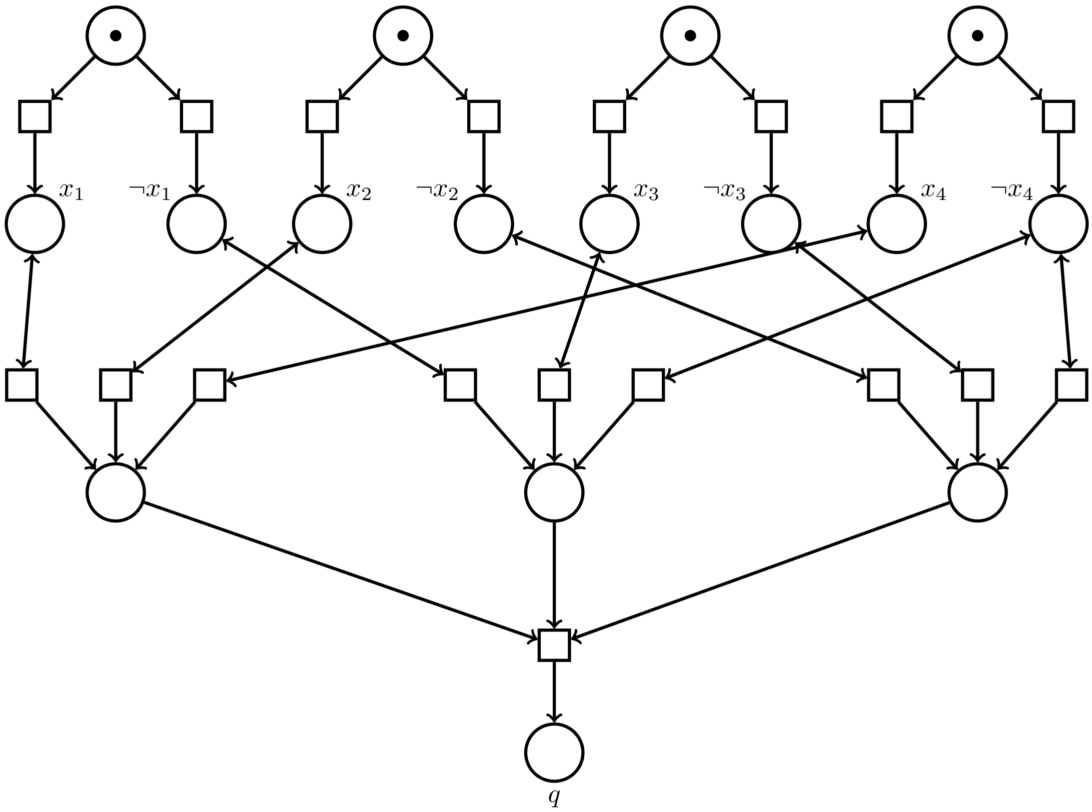
1. Add the following transitions to $\mathcal{N}$: t such that ${}^\bullet t = M$ and $t^\bullet = \{s_1, s_2\}$, and t_i such that ${}^\bullet t_i = \{s_i\}$ and $t^\bullet = \emptyset$ for $i = 1, 2$. Transitions t_1 and t_2 are simultaneously enabled if and only if marking M is coverable in $(\mathcal{N}, M_0)$.
2. Add a transition t to $\mathcal{N}$ such that ${}^\bullet t = M$ and $t^\bullet = \emptyset$. Transition t is enabled if and only if marking M is coverable in $(\mathcal{N}, M_0)$.
3. Add a transition t to $\mathcal{N}$ such that ${}^\bullet t = M$ and $t^\bullet = M$. If t is enabled in some marking M' reachable from M_0, then $M' \geq {}^\bullet t = M$ so M is coverable in $(\mathcal{N}, M_0)$. If M is coverable then there exists a firing sequence σ such that $M_0 \xrightarrow{\sigma} M' \geq M$. The run $\sigma ttt(t)^\omega$ is a run which contains t infinitely often.

(b) Let $(\mathcal{N}, M_0)$ be a 1-bounded Petri net.

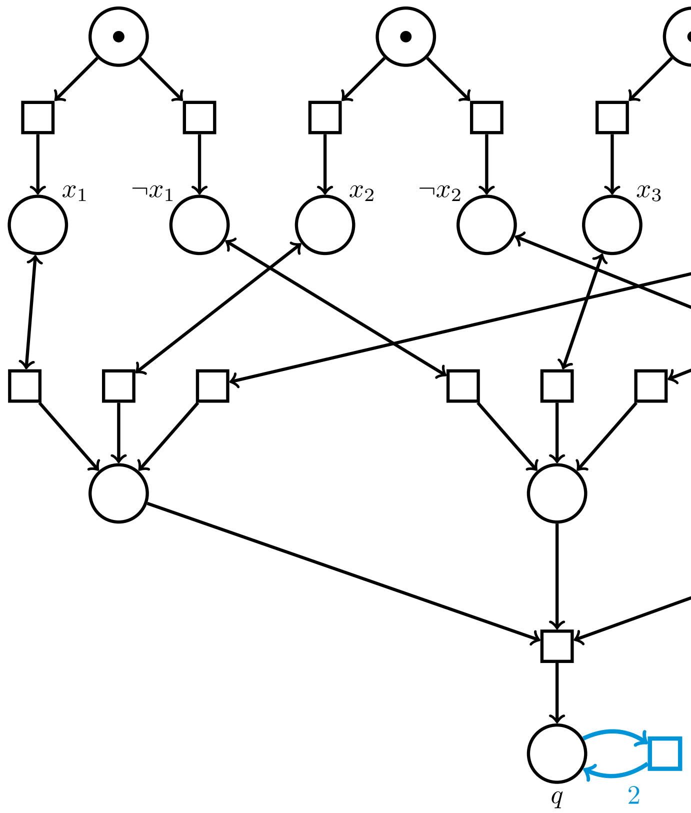
1. Let M be the marking equal to the multiset addition of ${}^\bullet t_1 + {}^\bullet t_2$. Transitions t_1 and t_2 are simultaneously enabled if and only if marking M is coverable in $(\mathcal{N}, M_0)$.
2. Let M be the marking equal to the multiset ${}^\bullet t$. Transition t is enabled if and only if marking M is coverable in $(\mathcal{N}, M_0)$.

Solution 3.4

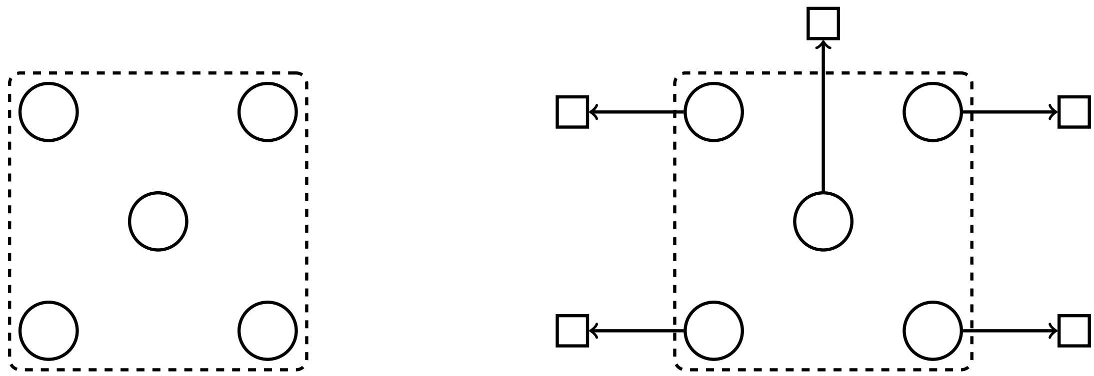
(a)



(b)



(c) ★ Given a Petri net, we make it *lossy* by adding, for each place p, a transition s_p that consumes a token from p:



original Petri net   transformed Petri net

More formally, given a Petri net with weighted arcs $\mathcal{N} = (P, T, W)$, we build the Petri net $\mathcal{N}' = (P, T', W')$ where

$$T' = T \cup \{s_p : p \in P\},$$

$$W'(t, p) = \begin{cases} W(t, p) & \text{if } t \in T, \\ 0 & \text{otherwise.} \end{cases}$$

$$W'(p, t) = \begin{cases} W(p, t) & \text{if } t \in T, \\ -1 & \text{if } t = s_p, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for every $M, M' \in \mathbb{N}^P$, M' is coverable in $(\mathcal{N}, M)$ if and only if M' is reachable in $(\mathcal{N}', M)$.

(d) ★ We prove the above claim. Let $M, M' \in \mathbb{N}^P$.

$\Rightarrow$) Suppose that M' is coverable in $(\mathcal{N}, M)$. There exist $M'' \in \mathbb{N}^P$ and $\sigma \in T^*$ such that $M'' \geq M'$ and $M \xrightarrow{\sigma} M''$ in $\mathcal{N}$. This implies that $M \xrightarrow{\sigma} M''$ in $\mathcal{N}'$. Since $M'' \geq M'$, we have $M'' \xrightarrow{*} M'$ in $\mathcal{N}'$ by decreasing the number of tokens accordingly. Therefore, $M \xrightarrow{*} M'' \xrightarrow{*} M'$ in $\mathcal{N}'$.

$\Leftarrow$) Suppose that M' is reachable in $(\mathcal{N}', M)$. There exists $\sigma \in (T')^*$ such that $M \xrightarrow{\sigma} M'$. By definition of $\mathcal{N}'$, for every $t \in T' \setminus T$ and $p \in P$, we have $W(t, p) = 0$. Thus, by #3.2(c), all transitions of $T' \setminus T$ occurring in σ can be moved to the end. More formally, there exists $\pi \in T^*$, $\pi' \in (T' \setminus T)^*$ and $M'' \in \mathbb{N}^P$ such that $\sigma = \pi\pi'$ and

$$M \xrightarrow{\pi} M'' \xrightarrow{\pi'} M'.$$

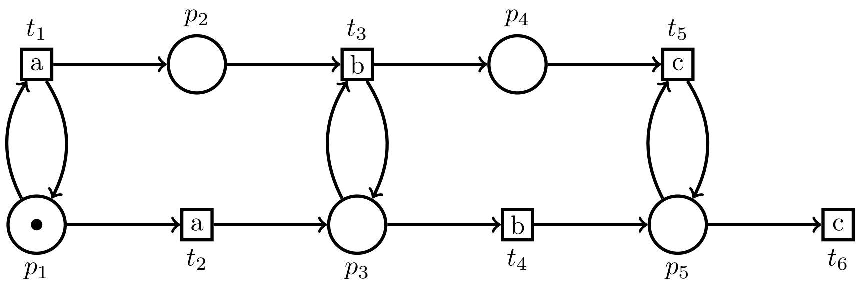
Since π' does not produce any token, we have $M'' \geq M'$. Moreover, $M \xrightarrow{\pi} M''$ is a firing sequence of $\mathcal{N}$ since $\pi \in T^*$. Therefore, M'' is coverable in $(\mathcal{N}, M)$. □

Solution 3.5

(a) Let L be a regular language given by a DFA (deterministic finite automaton) $\mathcal{A}$. Without loss of generality, we can take $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ with a single initial state q_0, transition function $\delta : Q \times \Sigma \rightharpoonup Q$ and final states $F \subseteq Q$.
We transform $\mathcal{A}$ into a terminal labeled Petri net $(\mathcal{N}, M_0, \mathcal{C}_t)$. The set of places of $\mathcal{N}$ is Q, its set of labels is Σ and M_0 is the marking that puts one token in q_0 and 0 elsewhere. For every transition $t : q \xrightarrow{a} q'$ in $\mathcal{A}$, we define a transition t in $\mathcal{N}$ such that $(q, t) \cup (t, q')$ is in the flow of $\mathcal{N}$ and $s_{\mathcal{N}}(t) = a$. We define the set of terminal markings $\mathcal{C}_t$ as the markings M_q that put one token in q and 0 elsewhere, for every final state $q \in F$.

(b) We display a terminal labeled Petri net $(\mathcal{N}, M_0, \mathcal{C}_t)$ whose terminal language is $\{a^n b^n c^n | n \in \mathbb{N}, n \geq 1\}$.

Let $(\mathcal{N}, M_0)$ be the labeled Petri net illustrated below, with $M_0 = (1, 0, 0, 0, 0)$. Let $\mathcal{C}_t$ be the singleton set $\{(0, 0, 0, 0, 0)\}$.



This petri net appears in [1, Abb. 6.1].

(c) We reason by contradiction. Let us assume L is a terminal language for a terminal free labeled Petri net $(\mathcal{N}, M_0, \mathcal{C}_t)$. Since a free labeled net labels transitions uniquely, we must have at least three transitions a, b, c in $\mathcal{N}$. Transition sequences ab and ba are both enabled from M_0 because abc and ba are in the language. The monotonicity lemma for regular Petri nets still holds for labeled Petri nets, as labelling the transitions does not modify the proof of this lemma. Therefore initial marking M_0 is modified in the same way by the action of transition sequences ab and ba. That is, there exists M such that $M_0 \xrightarrow{ab} M$ and $M_0 \xrightarrow{ba} M$. Since ba is in the language, M must be a terminal marking; but then ab is also in L, and this is not the case.

References

[1] Lutz Priese and Harro Wimmel. Petri-netze. ISBN 3-540-44289-8, 2003.