An SMT-based Approach to Fair Termination Analysis

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Fair Termination Analysis

- Fair termination: No non-fair infinite execution sequence σ.
- PSPACE-complete for boolean programs.

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SMT-Based Approach

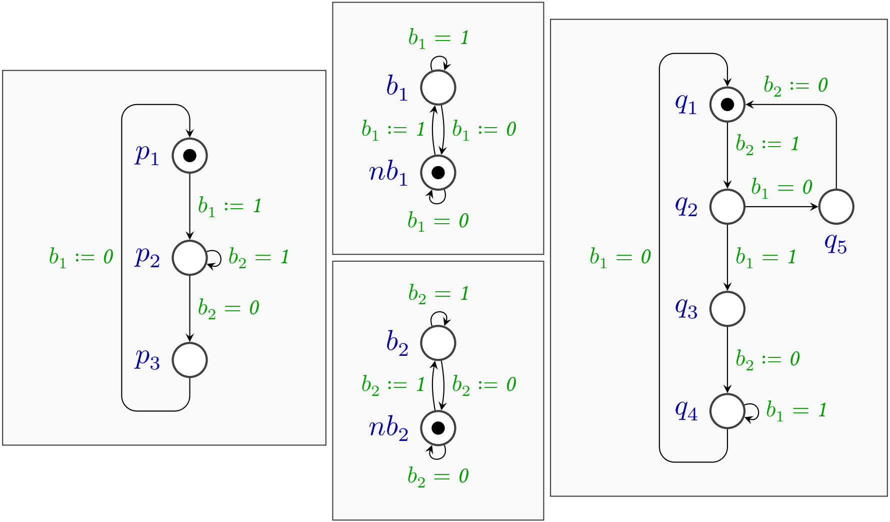
- Incomplete method based on reduction to feasibility of linear arithmetic constraints.
- Strengthened with refinement cycle which adds mixed linear and boolean constraints.
- Similar method previously applied for safety properties (An SMT-based Approach to Coverability Analysis, CAV14).

Lamport's 1-bit Algorithm for Mutual Exclusion

```
    procedure Process 1
    begin
      b1 := 0
      while true do
p1:     b1 := 1
p2:     while b2 = 1 do skip od
p3:     (* critical section *)
        b1 := 0
      od
    end
```

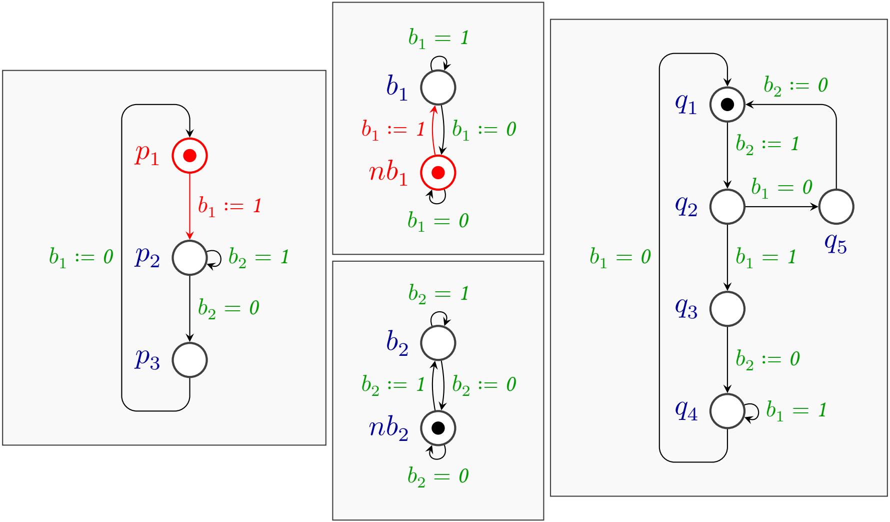
```
    procedure Process 2
    begin
      b2 := 0
      while true do
q1:     b2 := 1
q2:     if b1 = 1 then
q3:       b2 := 0
q4:       while b1 = 1 do skip od
          goto q1
        fi
q5:     (* critical section *)
        b2 := 0
      od
    end
```

Communicating Automata Model



Property: If both processes are executed infinitely often, then the first process should enter the critical section (p_3) infinitely often.

Communicating Automata Model



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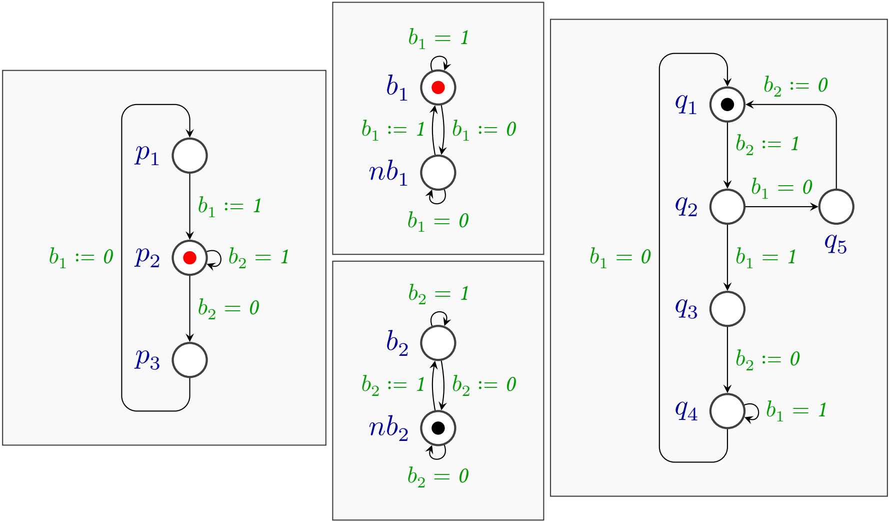
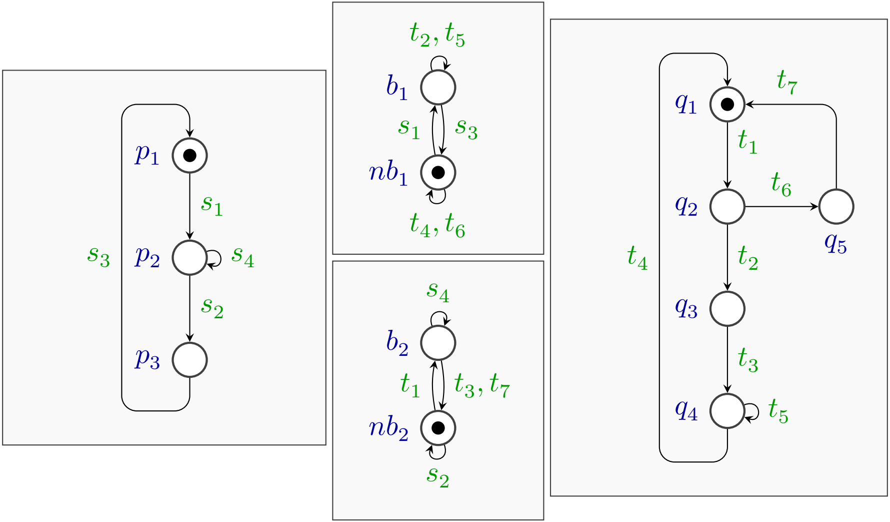


Property: If both processes are executed infinitely often, then the first process should enter the critical section (p_3) infinitely often.

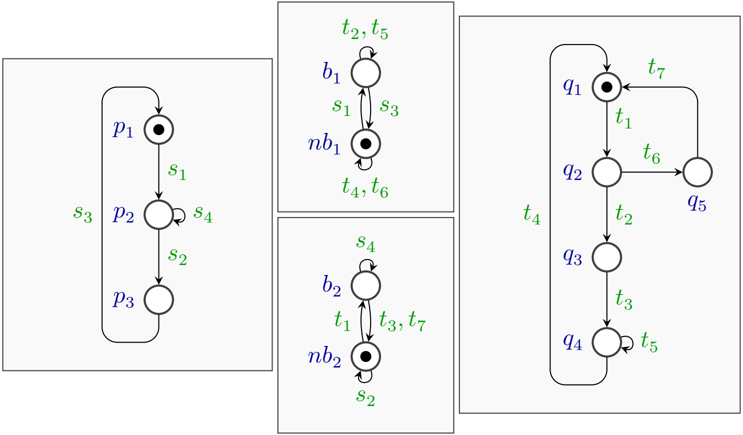
Abstract View of the Model



Property: For every infinite transition sequence σ, we have
$\varphi(\sigma) = \bigvee_{i=1}^{4}(s_i \in \inf(\sigma)) \wedge \bigvee_{i=1}^{7}(t_i \in \inf(\sigma)) \implies s_2 \in \inf(\sigma)$.

Loop Sequences

$$\{p_1, nb_1, nb_2, q_1\} \xrightarrow{t_1 t_6 t_7 s_1 t_1 t_2 t_3 s_2 t_5 s_3 t_4} \{p_1, nb_1, nb_2, q_1\}$$



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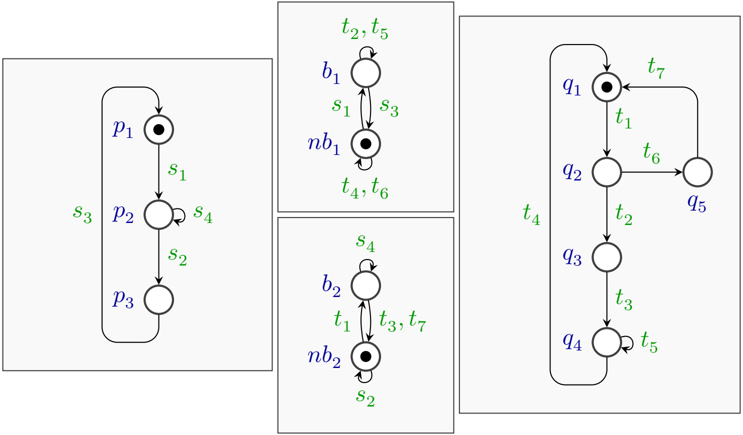
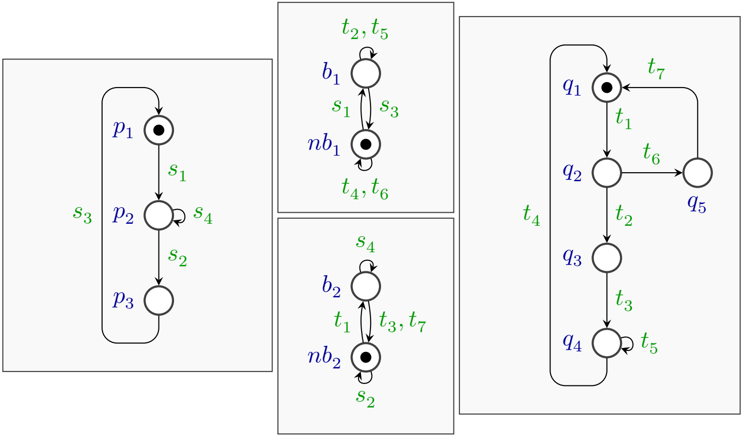
$$\#\sigma = \begin{pmatrix} \#t_1 & \#t_2 & \#t_3 & \#t_4 & \#t_5 & \#t_6 & \#t_7 & \#s_1 & \#s_2 & \#s_3 & \#s_4 \\ & & & & & & & & & & \end{pmatrix}$$



Loop Sequences

$$\{p_1, nb_1, nb_2, q_1\} \xrightarrow{t_1 t_6 t_7 s_1 t_1 t_2 t_3 s_2 t_5 s_3 t_4} \{p_1, nb_1, nb_2, q_1\}$$

	$\#t_1$	$\#t_2$	$\#t_3$	$\#t_4$	$\#t_5$	$\#t_6$	$\#t_7$	$\#s_1$	$\#s_2$	$\#s_3$	$\#s_4$	
$\#\sigma = ($	2											$)$



Loop Sequences

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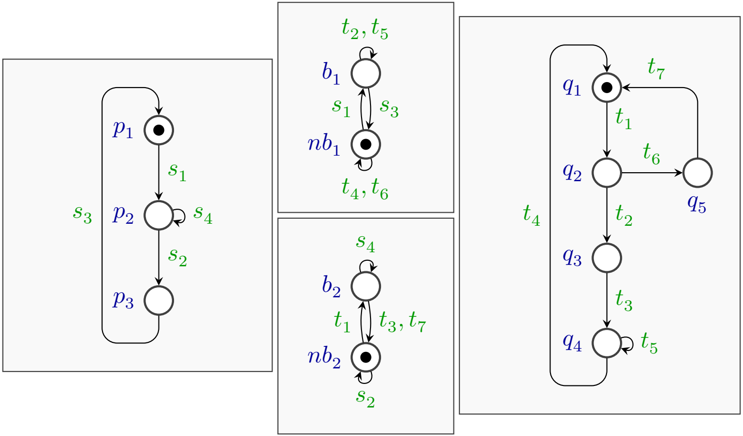
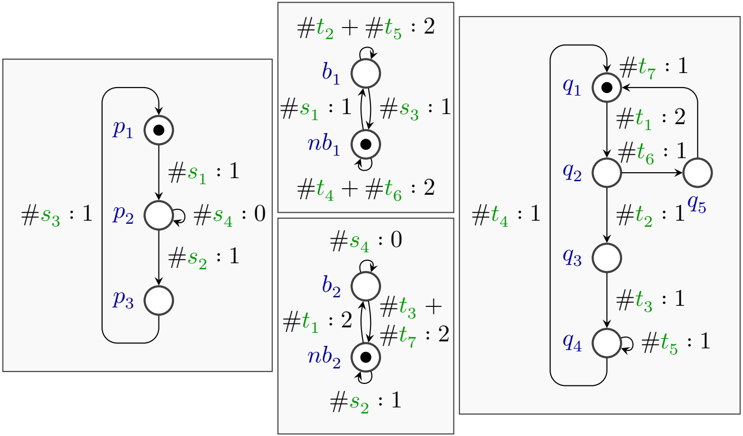
$$\#\sigma = \begin{pmatrix} \#t_1 & \#t_2 & \#t_3 & \#t_4 & \#t_5 & \#t_6 & \#t_7 & \#s_1 & \#s_2 & \#s_3 & \#s_4 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$



Loop Sequences

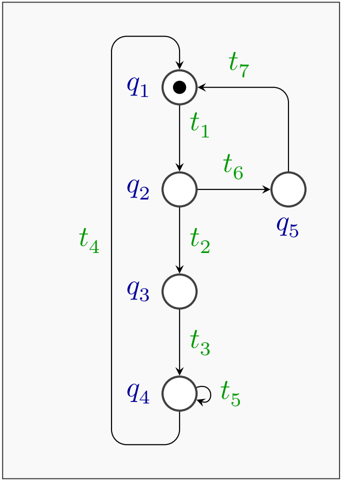
$$\{p_1, nb_1, nb_2, q_1\} \xrightarrow{t_1 t_6 t_7 s_1 t_1 t_2 t_3 s_2 t_5 s_3 t_4} \{p_1, nb_1, nb_2, q_1\}$$

	$\#t_1$	$\#t_2$	$\#t_3$	$\#t_4$	$\#t_5$	$\#t_6$	$\#t_7$	$\#s_1$	$\#s_2$	$\#s_3$	$\#s_4$
$\#\sigma =$	2	1	1	1	1	1	1	1	1	1	0



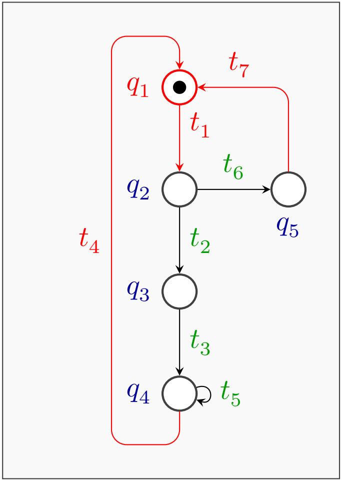
Necessary Condition for Loops

$$X = \begin{pmatrix} \overset{\#t_1}{t_1} & \overset{\#t_2}{t_2} & \overset{\#t_3}{t_3} & \overset{\#t_4}{t_4} & \overset{\#t_5}{t_5} & \overset{\#t_6}{t_6} & \overset{\#t_7}{t_7} & \overset{\#s_1}{s_1} & \overset{\#s_2}{s_2} & \overset{\#s_3}{s_3} & \overset{\#s_4}{s_4} \end{pmatrix}$$



Necessary Condition for Loops

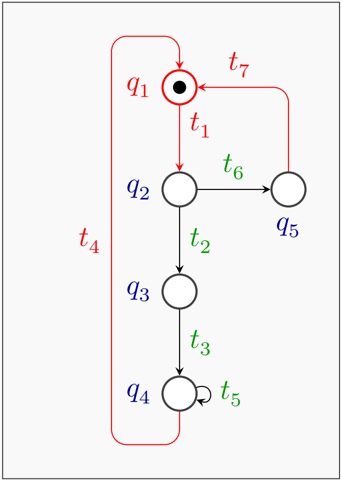
$$X = \begin{pmatrix} \#t_1 & \#t_2 & \#t_3 & \#t_4 & \#t_5 & \#t_6 & \#t_7 & \#s_1 & \#s_2 & \#s_3 & \#s_4 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & s_1 & s_2 & s_3 & s_4 \end{pmatrix}$$



Necessary Condition for Loops

$$X = \begin{pmatrix} \overset{\#t_1}{t_1} & \overset{\#t_2}{t_2} & \overset{\#t_3}{t_3} & \overset{\#t_4}{t_4} & \overset{\#t_5}{t_5} & \overset{\#t_6}{t_6} & \overset{\#t_7}{t_7} & \overset{\#s_1}{s_1} & \overset{\#s_2}{s_2} & \overset{\#s_3}{s_3} & \overset{\#s_4}{s_4} \end{pmatrix}$$

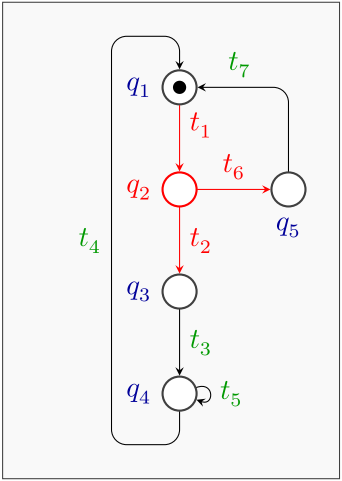
$q_1 : \quad t_4 + t_7 = t_1$



Necessary Condition for Loops

$$X = \begin{pmatrix} \#t_1 & \#t_2 & \#t_3 & \#t_4 & \#t_5 & \#t_6 & \#t_7 & \#s_1 & \#s_2 & \#s_3 & \#s_4 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & s_1 & s_2 & s_3 & s_4 \end{pmatrix}$$

$q_1 : \quad t_4 + t_7 = t_1$

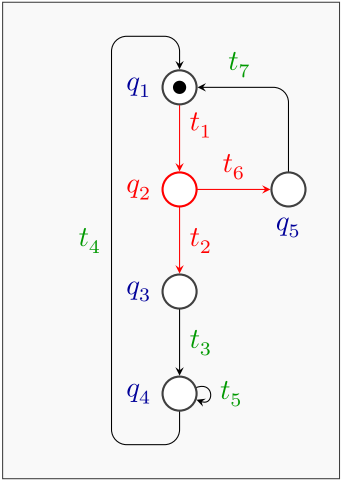


Necessary Condition for Loops

$$X = \begin{pmatrix} \#t_1 & \#t_2 & \#t_3 & \#t_4 & \#t_5 & \#t_6 & \#t_7 & \#s_1 & \#s_2 & \#s_3 & \#s_4 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & s_1 & s_2 & s_3 & s_4 \end{pmatrix}$$

$$q_1: \quad t_4 + t_7 = t_1$$

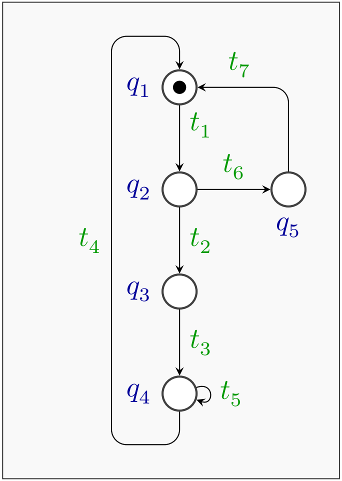
$$q_2: \quad t_1 = t_2 + t_6$$



Necessary Condition for Loops

$$X = \begin{pmatrix} \#t_1 & \#t_2 & \#t_3 & \#t_4 & \#t_5 & \#t_6 & \#t_7 & \#s_1 & \#s_2 & \#s_3 & \#s_4 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & s_1 & s_2 & s_3 & s_4 \end{pmatrix}$$

$$\begin{aligned} q_1 &: & t_4 + t_7 &= t_1 \\ q_2 &: & t_1 &= t_2 + t_6 \\ q_3 &: & t_2 &= t_3 \\ q_4 &: & t_3 &= t_4 \\ q_5 &: & t_6 &= t_7 \end{aligned}$$



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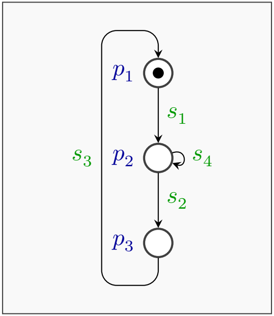
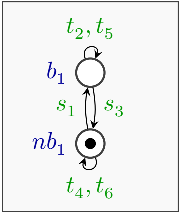
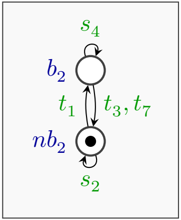
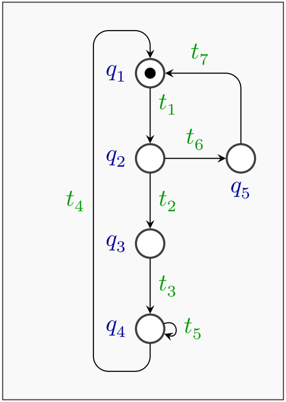








$$p_1: \quad s_3 = s_1$$
$$p_2: \quad s_1 = s_2$$
$$p_3: \quad s_2 = s_3$$

$$b_1: \quad s_1 = s_3$$
$$nb_1: \quad s_3 = s_1$$

$$b_2: \quad t_1 = t_3 + t_7$$
$$nb_2: \quad t_3 + t_7 = s_1$$

$$q_1: \quad t_4 + t_7 = t_1$$
$$q_2: \quad t_1 = t_2 + t_6$$
$$q_3: \quad t_2 = t_3$$
$$q_4: \quad t_3 = t_4$$
$$q_5: \quad t_6 = t_7$$

Termination Constraints

- Accumulate constraints in matrix form as $C \cdot X = 0$.
- If there is an infinite transition sequence σ, then the following constraints have a solution X:

$$\mathcal{C} :: \begin{cases} C \cdot X = 0 \\ X \geq 0 \\ X \neq 0 \end{cases}$$

- If the constraints have no solution, then the program is terminating.
- A solution X is *realizable* if there is a sequence σ with $\#\sigma = X$.

Fair Termination Constraints

- Fairness condition given by boolean formula φ over $t \in \inf(\sigma)$.
- If the program is not fairly terminating, then there is an infinite transition sequence σ satisfying $\sigma \models \neg\varphi$.
- Add constraint $\neg\varphi(X)$ to $\mathcal{C}$ for fair termination constraints.

Fairness for Lamport's Algorithm

$$\varphi(\sigma) = \bigvee_{i=1}^{4}(s_i \in \inf(\sigma)) \wedge \bigvee_{i=1}^{7}(t_i \in \inf(\sigma)) \implies s_2 \in \inf(\sigma)$$

$$\begin{aligned}\neg\varphi(X) = \; & (s_1 + s_2 + s_3 + s_4 > 0) \wedge \\ & (t_1 + t_3 + t_4 + t_5 + t_6 + t_7 > 0) \wedge \\ & (s_2 = 0)\end{aligned}$$

Fair Termination Constraints

$$
\begin{array}{llll}
s_3 = s_1 & t_4 + t_7 = t_1 & s_1 \geq 0 & t_1 \geq 0 \\
s_1 = s_2 & t_1 = t_2 + t_6 & s_2 \geq 0 & t_2 \geq 0 \\
s_2 = s_3 & t_2 = t_3 & s_3 \geq 0 & t_3 \geq 0 \\
 & t_3 = t_4 & & t_4 \geq 0 \\
 & t_6 = t_7 & & t_5 \geq 0 \\
s_1 = s_3 & t_1 = t_3 + t_7 & & t_6 \geq 0 \\
s_3 = s_1 & t_3 + t_7 = s_1 & & t_7 \geq 0
\end{array}
$$

$$s_1 + s_2 + s_3 + s_4 + t_1 + t_3 + t_4 + t_5 + t_6 + t_7 > 0$$

$$
\begin{array}{l}
(s_1 + s_2 + s_3 + s_4 > 0) \wedge \\
(t_1 + t_3 + t_4 + t_5 + t_6 + t_7 > 0) \wedge \\
(s_2 = 0)
\end{array}
$$

Fair Termination Constraints: Solution

$$X = \begin{pmatrix} \overset{t_1}{0} & \overset{t_2}{0} & \overset{t_3}{0} & \overset{t_4}{0} & \overset{t_5}{1} & \overset{t_6}{0} & \overset{t_7}{0} & \overset{s_1}{0} & \overset{s_2}{0} & \overset{s_3}{0} & \overset{s_4}{1} \end{pmatrix}$$

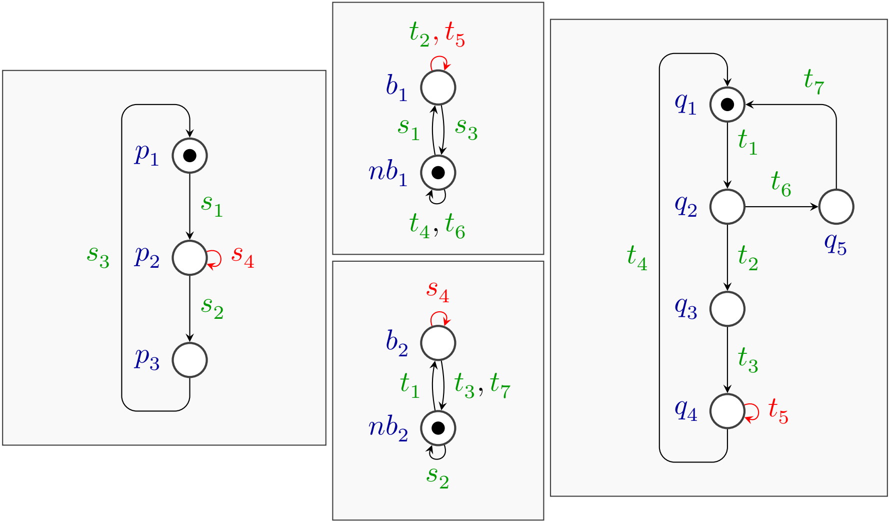
$$\begin{array}{llll}
s_3 = s_1 & t_4 + t_7 = t_1 & s_1 \geq 0 & t_1 \geq 0 \\
s_1 = s_2 & t_1 = t_2 + t_6 & s_2 \geq 0 & t_2 \geq 0 \\
s_2 = s_3 & t_2 = t_3 & s_3 \geq 0 & t_3 \geq 0 \\
 & t_3 = t_4 & & t_4 \geq 0 \\
 & t_6 = t_7 & & t_5 \geq 0 \\
s_1 = s_3 & t_1 = t_3 + t_7 & & t_6 \geq 0 \\
s_3 = s_1 & t_3 + t_7 = s_1 & & t_7 \geq 0
\end{array}$$

$$s_1 + s_2 + s_3 + s_4 + t_1 + t_3 + t_4 + t_5 + t_6 + t_7 > 0$$

$$(s_1 + s_2 + s_3 + s_4 > 0) \wedge$$
$$(t_1 + t_3 + t_4 + t_5 + t_6 + t_7 > 0) \wedge$$
$$(s_2 = 0)$$

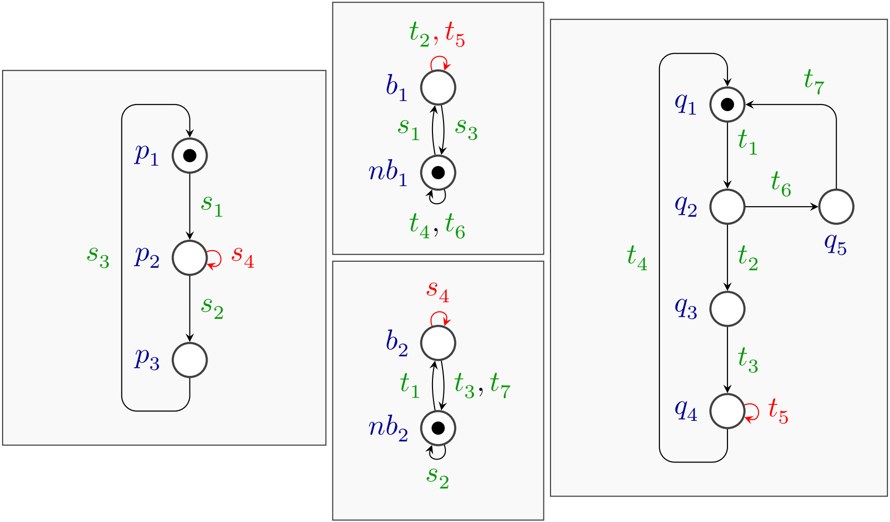
Fair Termination Constraints: Solution

$$X = \begin{pmatrix} \overset{t_1}{0} & \overset{t_2}{0} & \overset{t_3}{0} & \overset{t_4}{0} & \overset{t_5}{1} & \overset{t_6}{0} & \overset{t_7}{0} & \overset{s_1}{0} & \overset{s_2}{0} & \overset{s_3}{0} & \overset{s_4}{1} \end{pmatrix}$$



Solution realizable?

X realized by σ with $\inf(\sigma) = \{s_4, t_5\}$.



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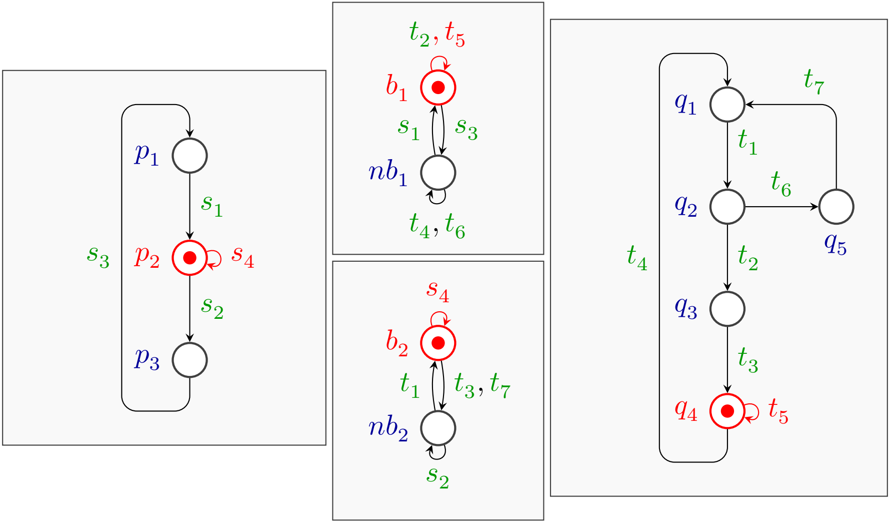
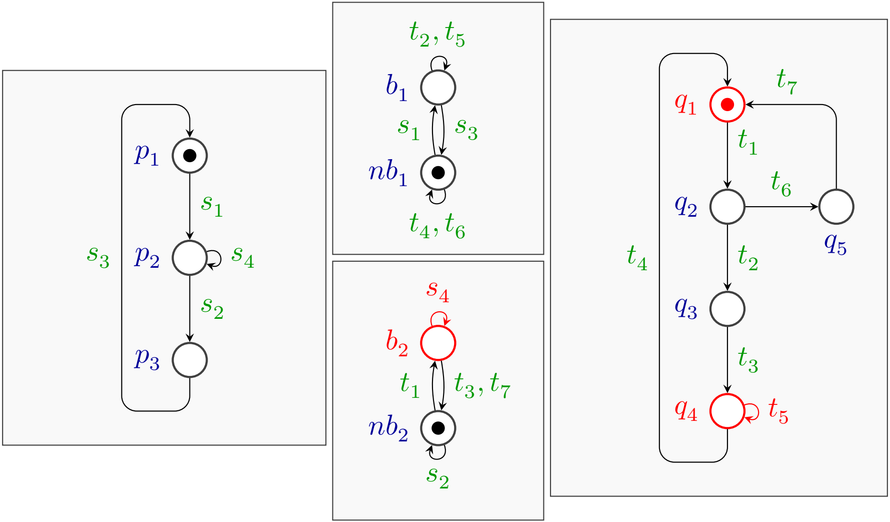


Refinement Component

q_1, q_4 and b_2 are in mutual exclusion.



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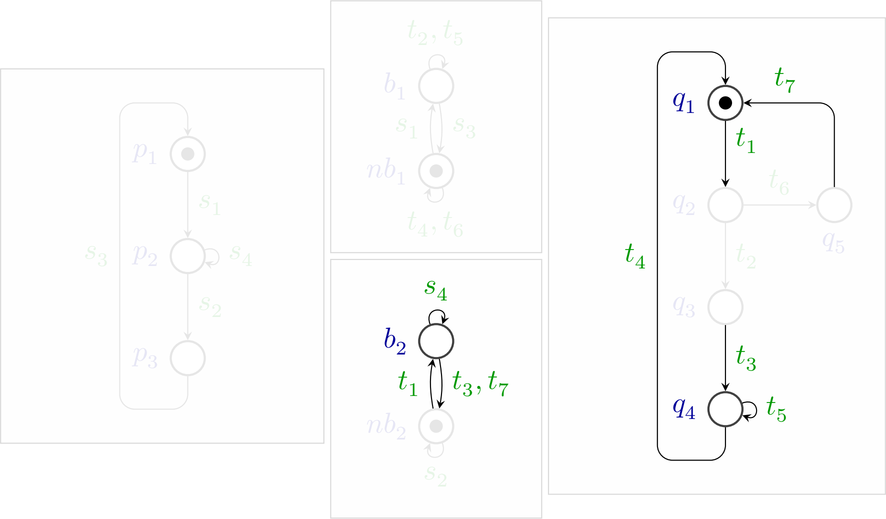

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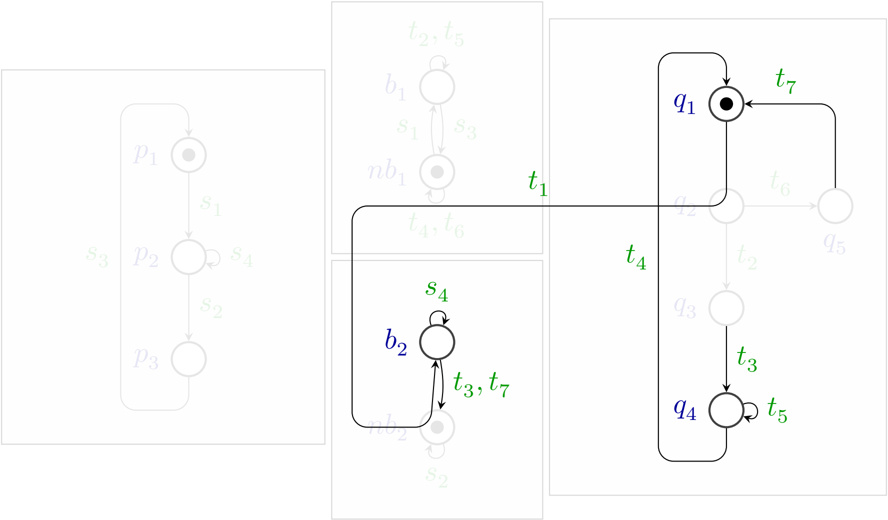

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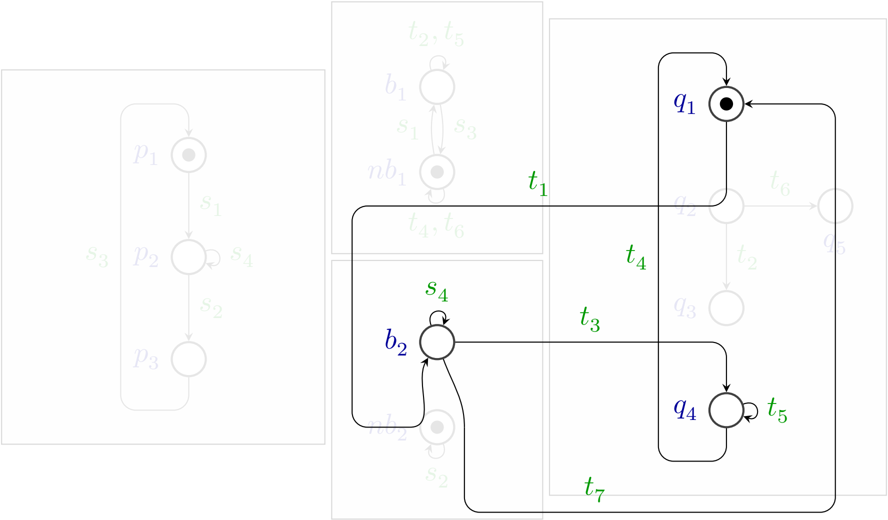
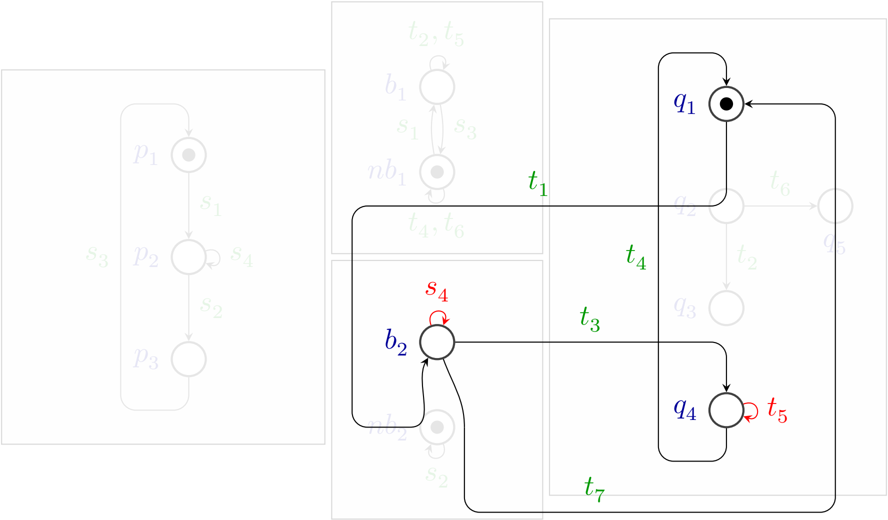


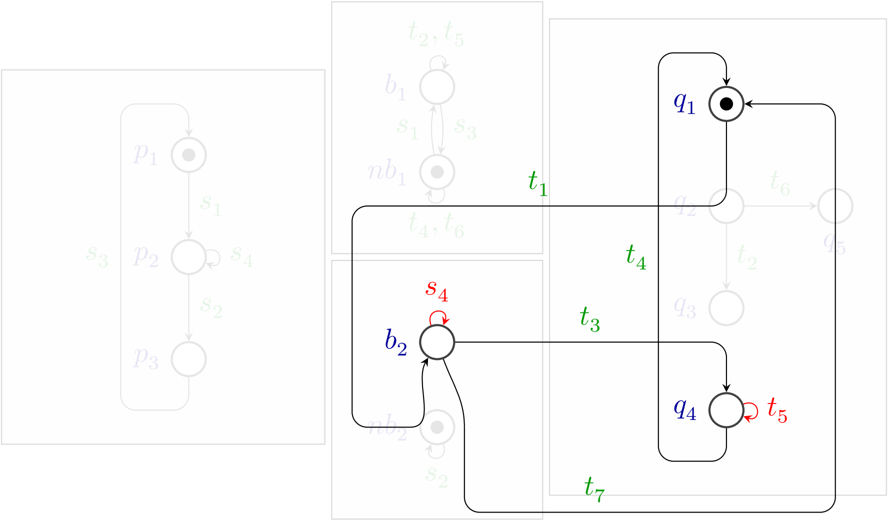
Refinement Constraint

X realized by σ with $\inf(\sigma) = \{s_4, t_5\}$.



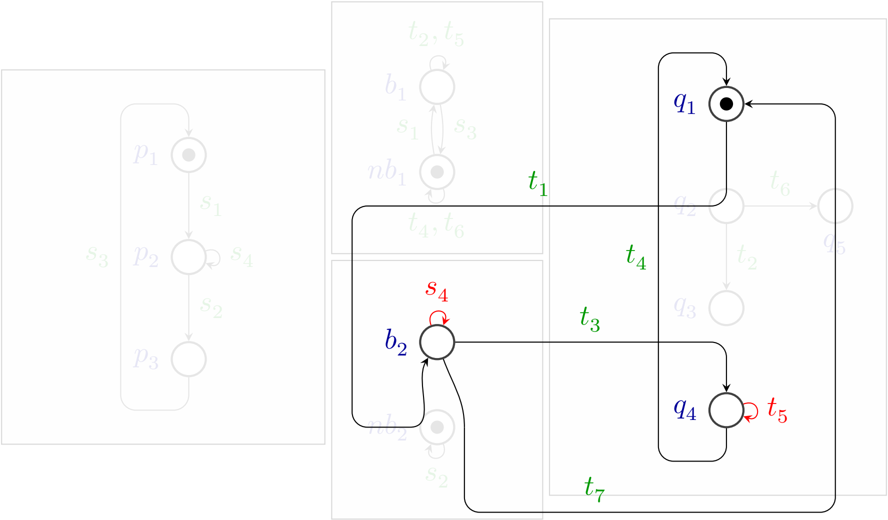
Refinement Constraint

X not realizable $\Rightarrow$ Generate refinement constraint δ.



Refinement Constraint

$$\delta = (s_4 = 0) \vee (t_5 = 0) \vee (t_1 + t_3 + t_4 + t_7 > 0)$$



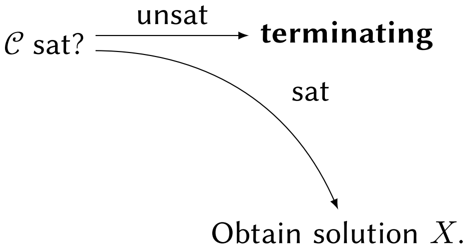
Refinement Loop

$\mathcal{C}$ sat?

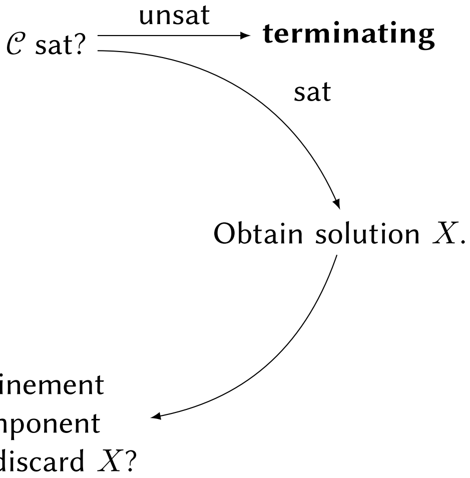
Refinement Loop

$\mathcal{C}$ sat? $\xrightarrow{\text{unsat}}$ **terminating**

Refinement Loop



Refinement Loop



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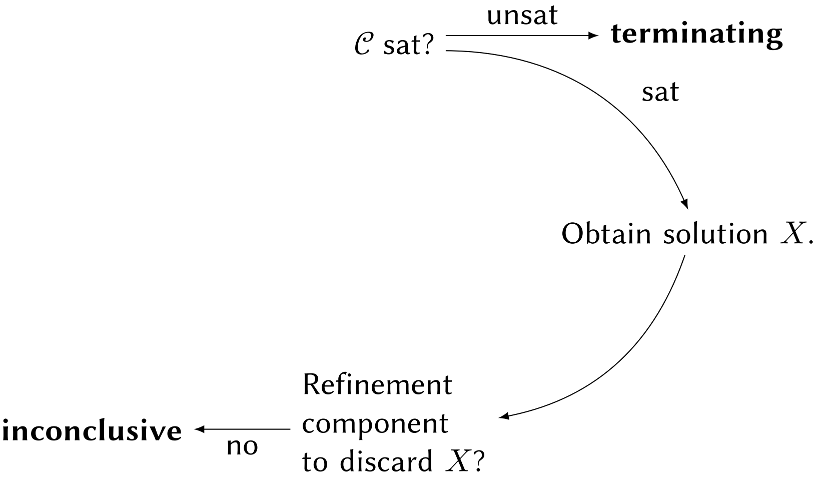

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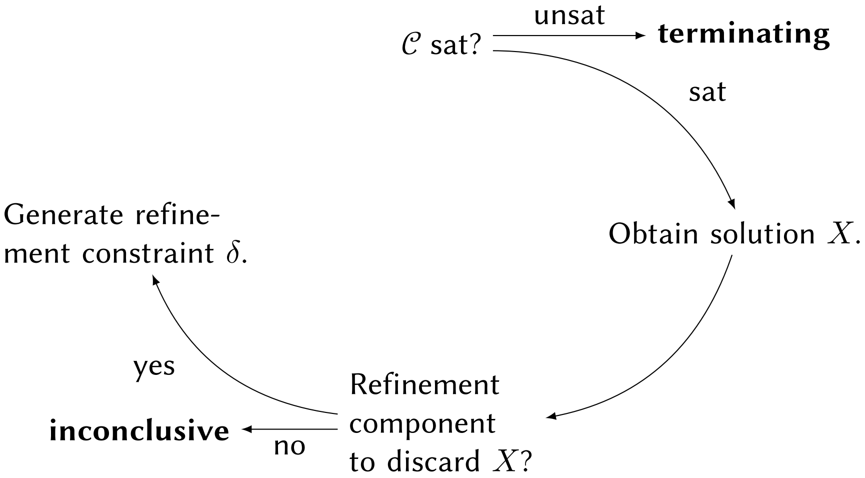
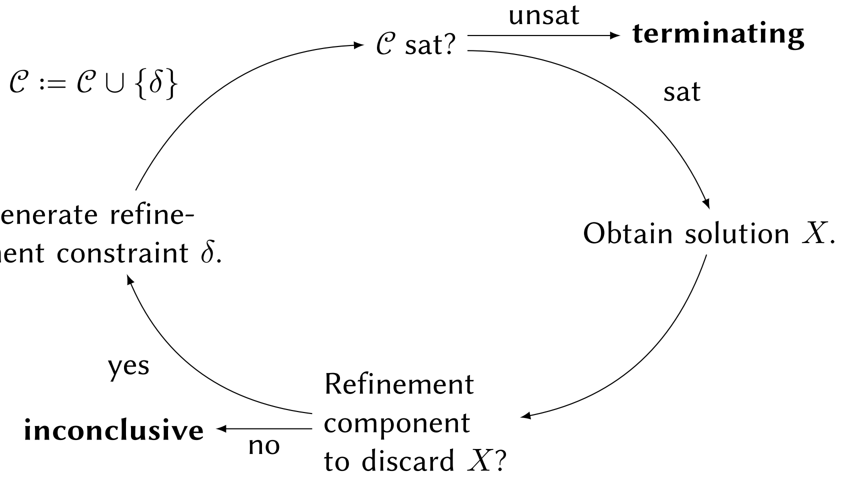


Refinement Loop

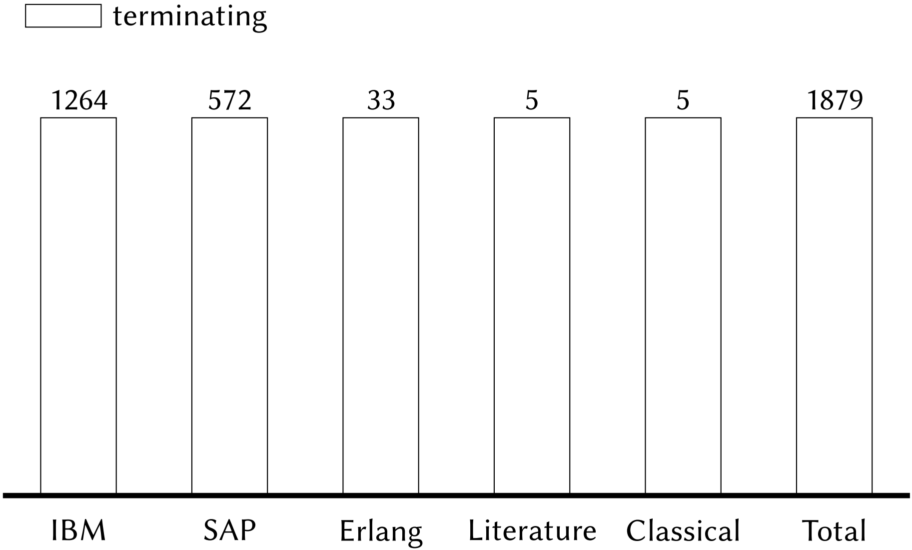


Experimental Evaluation

Benchmarks

- IBM/SAP — Workflow nets from business process models
  - 1976 examples
  - 1836 terminating
- Erlang — Models from the verification of Erlang programs
  - 50 examples, up to 66950 places and 213626 transitions
  - 33 terminating
- Literature — Selected examples from the literature
  - 5 examples, with unbounded variables
  - All terminating
- Classical — Classic asynchronous programs for mutual exclusion and distributed algorithms
  - 5 examples, scalable in number of processes
  - All fairly terminating

Rate of Success



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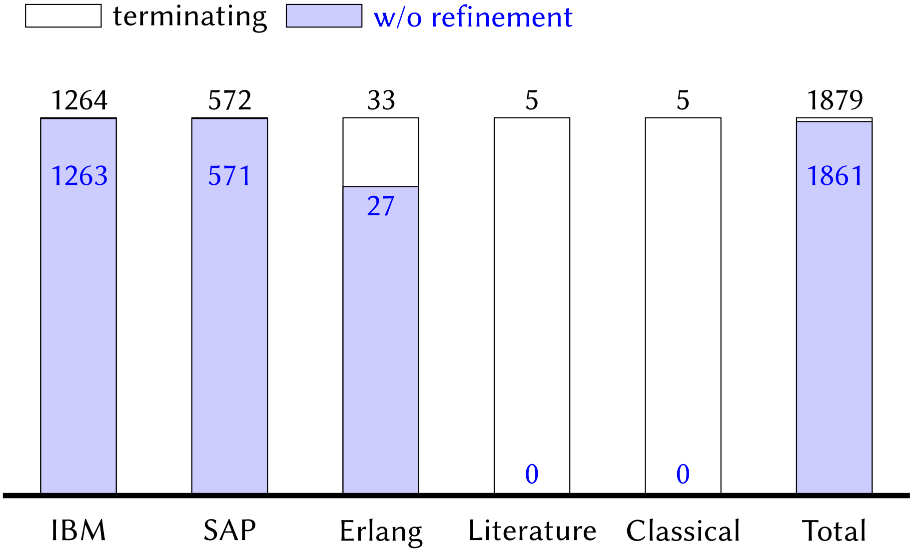
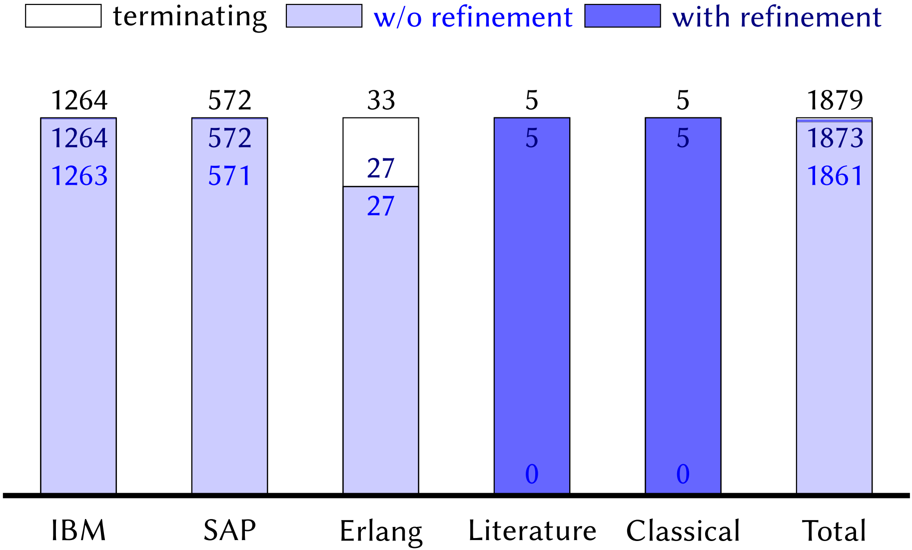
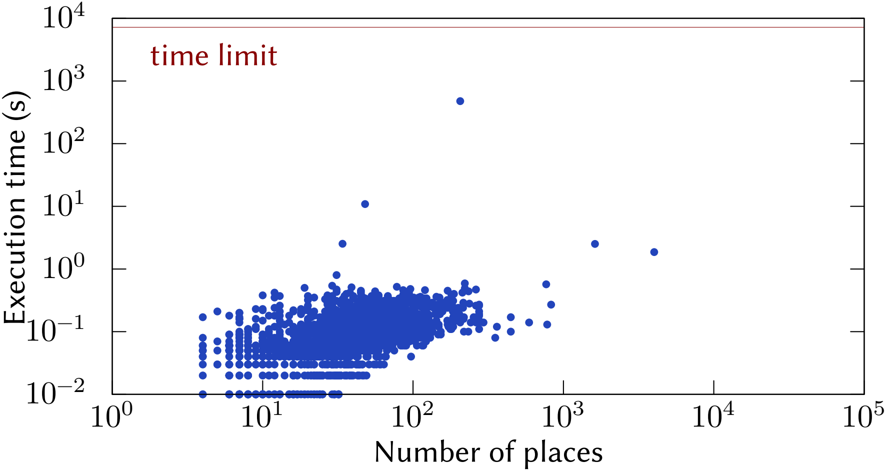


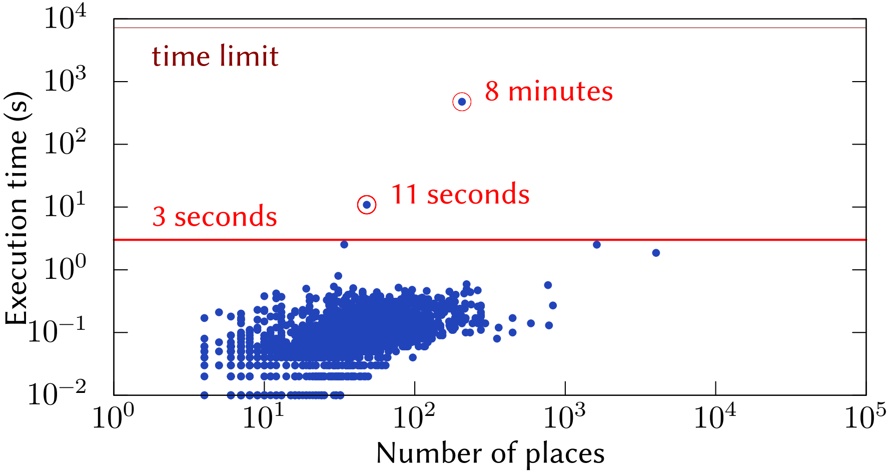
Rate of Success



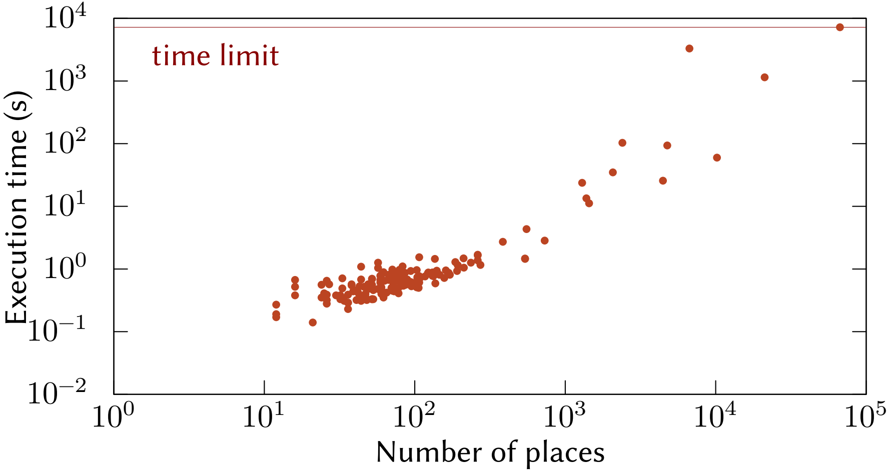
Performance on Positive Examples



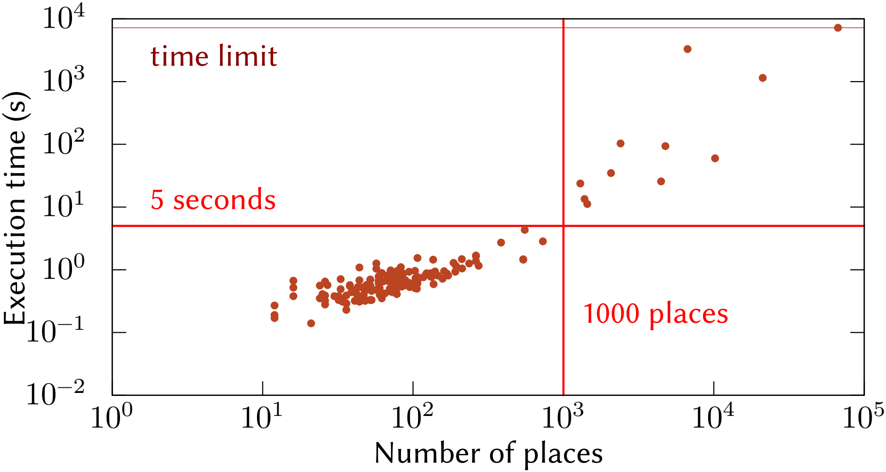
Performance on Positive Examples



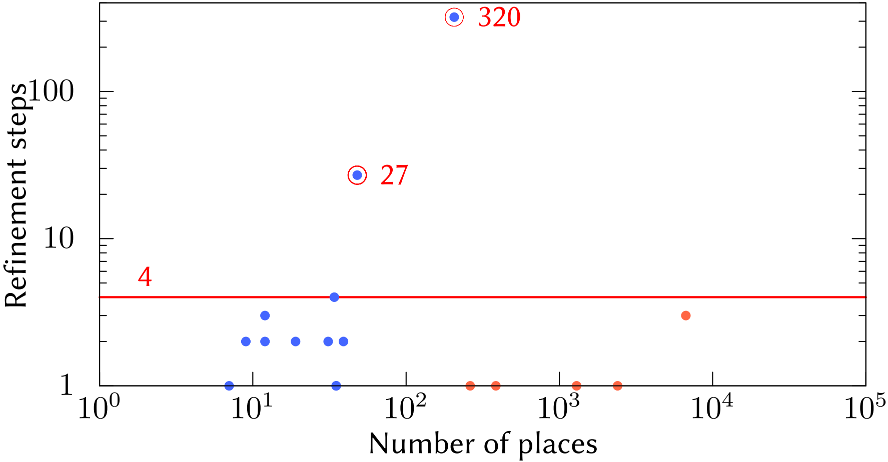
Performance on Negative Examples



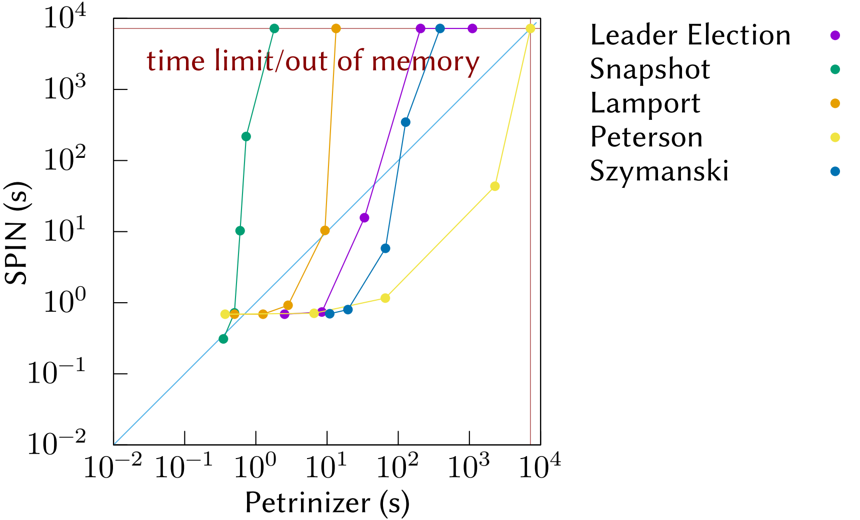
Performance on Negative Examples



Refinement Steps



Comparison with SPIN on Scaled Classical Suite



Summary

- Fast and effective technique for proving fair termination
- Incomplete, but high degree of completeness
- Large instances can be handled
- Constraints can be used as a certificate of fair termination