Shortest-path Theorem for T-systems

Lemma 1 Let (N, M_0) be a T-system and let $M_0 \xrightarrow{\sigma_1 \sigma_2 t}$ such that

- $t \notin \mathcal{A}(\sigma_1)$ ($\mathcal{A}(\sigma_1)$ is the set of transitions that occur in σ_1)
- $\mathcal{A}(\sigma_2) \subseteq \mathcal{A}(\sigma_1)$

Then $M_0 \xrightarrow{\sigma_1 t \sigma_2}$

Proof By induction on the length of σ_2

Base $|\sigma_2| = 0$ ✓

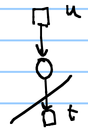
Step $\sigma_2 = \sigma_2' u$

We have to show $M_0 \xrightarrow{\sigma_1 t \sigma_2} \equiv M_0 \xrightarrow{\sigma_1 t \sigma_2' u}$

(a) We show $M_0 \xrightarrow{\sigma_1 \sigma_2' t u}$

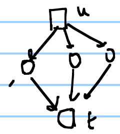
Two cases:

1) $u^\bullet \cap {}^\bullet t = \emptyset$



If t is enabled after u occurs, then it was also enabled before

2) $u^\bullet \cap {}^\bullet t \neq \emptyset$



$\sigma \sigma' u t$

$\sigma_1 \sigma_2 t u$

Since $u \in \mathcal{A}(\sigma_2)$ and $\mathcal{A}(\sigma_2) \subseteq \mathcal{A}(\sigma_1)$

we have $u \in \mathcal{A}(\sigma_1)$

So after $\sigma_1 \sigma_2' u$ there are at least 2 tokens in every place "between u and t" (because $t \notin \mathcal{A}(\sigma_1)$)

So t was already enabled before firing u, and so $\sigma_1 \sigma_2' t u$ is fireable.

(b) $M_0 \xrightarrow{\sigma_1 t \sigma_2' u}$

By (a) we have $M_0 \xrightarrow{\sigma_1 \sigma_2' t u}$

By IH since $|\sigma_2'| < |\sigma_2|$ we get

$M_0 \xrightarrow{\sigma_1 t \sigma_2' u}$

<u>Lemma 2</u> Let (N, M_0) be a 1-bounded T-system and let $M_0 \xrightarrow{\sigma} M$, $|\sigma| \geq 1$

Then there is σ_1, σ_2 such that

(a) $M_0 \xrightarrow{\sigma_1 \sigma_2} M$

(b) no transition occurs more than once in σ_1

(c) $\mathcal{A}(\sigma_2) \subset \mathcal{A}(\sigma_1)$

<u>Proof</u> First we prove the result with $\subseteq$ instead $\subset$ in (c).

By induction on $|\sigma|$

<u>Base</u> $|\sigma| = 1$. Take $\sigma_1 := \sigma$ and $\sigma_2 := \epsilon$

<u>Step</u> $\sigma = \tau t$, $\tau \neq \varepsilon$

By IH $M_0 \xrightarrow{\tau_1 \tau_2 t} M$ where

- no transition occurs more than once in τ_1
- $\mathcal{A}(\sigma_2) \subseteq \mathcal{A}(\sigma_1)$

- If $t \in \mathcal{A}(\sigma_1)$ then take $\sigma_1 = \tau_1$, $\sigma_2 = \tau_2 t$
- If $t \notin \mathcal{A}(\sigma_1)$ then by Lemma 1
  $M_0 \xrightarrow{\tau_1 t \tau_2} M_2$ and take $\sigma_1 = \tau_1 t$, $\sigma_2 = \tau_2$

- Now assume $\mathcal{A}(\sigma_1) = \mathcal{A}(\sigma_2)$
  - If $\mathcal{A}(\sigma_1)$ contains every transition then replace σ_1 by ε! (because $M_0 \xrightarrow{\sigma_1} M_0$)
  - If $\mathcal{A}(\sigma_1)$ does not contain every transition then the net is not 1-life, contradiction!!

  Ex: T-system $M_0 \xrightarrow{\sigma_1 \sigma_2}$
  $\mathcal{A}(\sigma_1)$ does not contain every transition
  $\mathcal{A}(\sigma_2) = \mathcal{A}(\sigma_1)$

Theorem Let (N, M_0) be a 1-safe T-system and let M be reachable from M_0.
Then $M_0 \xrightarrow{\sigma} M$ with $|\sigma| \leq \frac{n(n-1)}{2}$ where n is the number of transitions.

Proof By repeated application of Lemma 2 there is $\sigma_1 \sigma_2 \cdots \sigma_k$ such that

- $M_0 \xrightarrow{\sigma_1 \sigma_2 \cdots \sigma_k} M$
- no transition occurs more than once in σ_i for every $1 \leq i \leq k$
- $A(\sigma_i) \subset A(\sigma_{i+1})$ for every $1 \leq i \leq k-1$

It follows $|\sigma_1| \leq n$
$|\sigma_2| \leq n-1$ and $k \leq n$
$|\sigma_3| \leq n-2$
...

So $|\sigma_1 \cdots \sigma_k| \leq \sum_{i=1}^{n} (n-i) = \sum_{i=1}^{n} i = \frac{n(n-1)}{2}$

If (N, M_0) is live, then
$|\sigma_1 \cdots \sigma_k| = \sum_{i=1}^{n-1} i = \frac{(n-1)(n-2)}{2}$