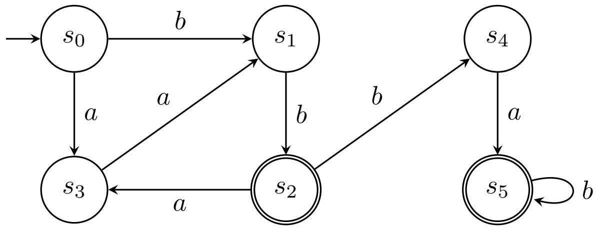
Automata and Formal Languages — Exercise Sheet 12

Exercise 12.1
Let B be the following Büchi automaton.



(a) For every state of B, give the discovery time and finishing time assigned by a DFS on B starting in s_0 (i.e. the moment they first become grey and the moment they become black). Visit successors s_i of a given state in the ascending order of their indices i. For example, when visiting the successors of s_2, first visit s_3 and later s_4.

(b) The language of B is not empty. Give the witness lasso found by applying $NestedDFS$ to B following the same convention for the order of successors as above.

(c) Is the execution in (b) optimal? Does there exists an optimal execution of $NestedDFS$ on B with a different order for visiting successors?

Exercise 12.2
Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. Give LTL formulas for the following ω-languages:

(a) $\Sigma \, \{p, q\} \, \{q\}^\omega$

(b) $\Sigma^* \, \{q\}^\omega$

(c) $\Sigma^* \, \emptyset \, \Sigma^\omega$

(d) $\{p\}^* \, \{p, q\} \, (\{p\} + \{p, q\})^\omega$

(e) $\{p\}^* \, \{q\}^* \, \emptyset^\omega$

Exercise 12.3
Let $\mathrm{AP} = \{p, q, r\}$. Give formulas for the computations satisfying the following properties:

(a) if q eventually holds, then p must not hold before q first does.

(b) if q eventually holds, then p holds at some point before q first holds.

(c) p always holds everywhere between q and r.

(d) p, and only p, holds at even positions and q, and only q, holds at odd positions.

Exercise 12.4
The *weak until* operator **W** has the following semantics:

- $\sigma \models \phi_1 \ \mathbf{W} \ \phi_2$ iff there exists $k \geq 0$ such that $\sigma^k \models \phi_2$ and $\sigma^i \models \phi_1$ for all $0 \leq i < k$, or $\sigma^k \models \phi_1$ for every $k \geq 0$.

Prove: $p \ \mathbf{W} \ q \equiv \mathbf{G}p \vee (p \ \mathbf{U} \ q) \equiv \mathbf{F}\neg p \rightarrow (p \ \mathbf{U} \ q) \equiv p \ \mathbf{U} \ (q \vee \mathbf{G}p)$.

Solution 12.1

a. We note "state[discovery time/finishing time]".
$s_0[1/12], s_1[2/11], s_2[3/10], s_3[4/5], s_4[6/9], s_5[7/8]$.

b. The lasso found by *NestedDFS* from s_0 is $s_0 s_1 s_2 s_3 s_4 s5s5$.

c. Given a non-empty NBA, we use the following definition of optimal execution of NestedDFS: the algorithm reports NONEMPTY at the earliest time such that all the states of a witness lasso have been explored.

   The execution given in (b) is non optimal since it does not return the lasso $s_0 s_1 s_2 s_3 s_1$ which already appeared in the explored subgraph.

   There is no execution of *NestedDFS* which blackens s_2 before s_5. But there is an execution of *NestedDFS* on B which returns the lasso $s_0 s_1 s_2 s_3 s_4 s5s5$ before it has visited the only other witness lasso $s_0 s_1 s_2 s_3 s_1$ and thus is optimal: the execution which does dfs1 via $s_0 s_1 s_2 s_4 s_5$, blackens s_5 then launches dfs2 from s_5 and finds a cycle. Node s_3 is not part of the explored subgraph so the algorithm reports NONEMPTY at the earliest time such that all the states of a witness lasso have been explored.

Solution 12.2

(a) $\mathbf{X}(p \wedge q) \wedge \mathbf{XXG}(\neg p \wedge q)$

(b) $\mathbf{FG}(\neg p \wedge q)$

(c) $\mathbf{F}(\neg p \wedge \neg q)$

(d) $\mathbf{G}p \wedge \mathbf{F}q$

(e) $(p \wedge \neg q)\ \mathbf{U}\ ((\neg p \wedge q)\ \mathbf{U}\ \mathbf{G}(\neg p \wedge \neg q))$

Solution 12.3

(a) $\mathbf{F}q \rightarrow (\neg p\ \mathbf{U}\ q)$

(b) $\mathbf{F}q \rightarrow (\neg q\ \mathbf{U}\ (\neg q \wedge p))$

(c) $\mathbf{G}((q \wedge \mathbf{XF}r) \rightarrow \mathbf{X}(p\ \mathbf{U}\ r))$

(d) $\mathbf{G}(\neg r) \wedge \mathbf{G}(p \leftrightarrow \neg q) \wedge p \wedge \mathbf{G}(p \rightarrow \mathbf{X}q) \wedge \mathbf{G}(q \rightarrow \mathbf{X}p)$

Solution 12.4

- $p\ \mathbf{W}\ q \equiv \mathbf{G}p \vee (p\ \mathbf{U}\ q)$.
  Follows immediately from the definitions.

- $\mathbf{G}p \vee (p\ \mathbf{U}\ q) \equiv \mathbf{F}\neg p \rightarrow (p\ \mathbf{U}\ q)$.
  We have: $\mathbf{G}p \vee (p\ \mathbf{U}\ q) \equiv \neg(\mathbf{F}\neg p) \vee (p\ \mathbf{U}\ q) \equiv \mathbf{F}\neg p \rightarrow (p\ \mathbf{U}\ q)$.

- $\mathbf{G}p \vee (p\ \mathbf{U}\ q) \equiv p\ \mathbf{U}\ (q \vee \mathbf{G}p)$.
  Assume $\sigma \models \mathbf{G}p \vee (p\ \mathbf{U}\ q)$ holds. If $\sigma \models \mathbf{G}p$, then $\sigma \models \varphi\ \mathbf{U}\ (\psi \vee \mathbf{G}p)$ for every φ, ψ. If $\sigma \models p\ \mathbf{U}\ q$, then $\sigma \models p\ \mathbf{U}\ (q \vee \psi)$ for every ψ.

  Assume $\sigma \models p\ \mathbf{U}\ (q \vee \mathbf{G}p)$. Then there exists $k \geq 0$ such that $\sigma^k \models q \vee \mathbf{G}p$ and $\sigma^i \models p$ for all $0 \leq i < k$. If $\sigma^k \models q$, then $\sigma \models p\ \mathbf{U}\ q$. If $\sigma^k \models \mathbf{G}p$, then $\sigma^i \models p$ for all $0 \leq i$, and so $\sigma \models \mathbf{G}p$. So $\sigma \models \mathbf{G}p \vee (p\ \mathbf{U}\ q)$.

  Or simply, using the fact given in the lecture that $\varphi\ \mathbf{U}\ (\psi_1 \vee \psi_2) \equiv (\varphi\ \mathbf{U}\ \psi_1) \vee (\varphi\ \mathbf{U}\ \psi_2)$, we have:
  $p\ \mathbf{U}\ (q \vee \mathbf{G}p) \equiv (p\ \mathbf{U}\ q) \vee (p\ \mathbf{U}\ \mathbf{G}p) \equiv (p\ \mathbf{U}\ q) \vee \mathbf{G}p \equiv \mathbf{G}p \vee (p\ \mathbf{U}\ q)$